Response Surface Methodology, Stat 579 — Homework 01 — Name: Answer Key
Fall 2014 — Prof. Erik B. Erhardt

Part I. (25 points) Refer to the "Background" reading for Chapter 1.

(5$^{\text{pts}}$) **1.** Refer to (71.) on page 12 of the reading. Assuming the proposed linear model is correct, verify that

$$(\underset{\sim}{y} - \mathbf{X}\underset{\sim}{b})^\top(\underset{\sim}{y} - \mathbf{X}\underset{\sim}{b}) = \underset{\sim}{y}^\top\underset{\sim}{y} - 2\underset{\sim}{b}^\top\mathbf{X}^\top\underset{\sim}{y} + \underset{\sim}{b}^\top\mathbf{X}^\top\mathbf{X}\underset{\sim}{b}.$$

Solution:

$$\begin{aligned}(\underset{\sim}{y} - \mathbf{X}\underset{\sim}{b})^\top(\underset{\sim}{y} - \mathbf{X}\underset{\sim}{b}) &= \underset{\sim}{y}^\top\underset{\sim}{y} - \underset{\sim}{y}^\top\mathbf{X}\underset{\sim}{b} - \underset{\sim}{b}^\top\mathbf{X}^\top\underset{\sim}{y} + \underset{\sim}{b}^\top\mathbf{X}^\top\mathbf{X}\underset{\sim}{b} \\ &= \underset{\sim}{y}^\top\underset{\sim}{y} - \underset{\sim}{y}^\top\mathbf{X}\underset{\sim}{b} - (\underset{\sim}{y}^\top\mathbf{X}\underset{\sim}{b})^\top + \underset{\sim}{b}^\top\mathbf{X}^\top\mathbf{X}\underset{\sim}{b} \\ &= \underset{\sim}{y}^\top\underset{\sim}{y} - 2\underset{\sim}{b}^\top\mathbf{X}^\top\underset{\sim}{y} + \underset{\sim}{b}^\top\mathbf{X}^\top\mathbf{X}\underset{\sim}{b}.\end{aligned}$$

Becasuse each of the quantities are scalars, $\underset{\sim}{b}^\top\mathbf{X}^\top\underset{\sim}{y} = \underset{\sim}{y}^\top\mathbf{X}\underset{\sim}{b}$.

(5$^{\text{pts}}$) **2.** Refer to (73.) on page 12 of the reading. Assuming the proposed linear model is correct, verify

$$\mathrm{E}[\underset{\sim}{b}] = \underset{\sim}{\beta}.$$

Solution:

$$\begin{aligned}\mathrm{E}[\underset{\sim}{b}] &= \mathrm{E}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{y}] \\ &= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathrm{E}[\underset{\sim}{y}] \\ &= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathrm{E}[\mathbf{X}\underset{\sim}{b} + \underset{\sim}{\varepsilon}] \\ &= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top(\mathbf{X}\underset{\sim}{\beta} + 0) \\ &= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}\underset{\sim}{\beta} \\ &= (\mathbf{X}^\top\mathbf{X})^{-1}(\mathbf{X}^\top\mathbf{X})\underset{\sim}{\beta} \\ &= \mathbf{I}\underset{\sim}{\beta} \\ &= \underset{\sim}{\beta}.\end{aligned}$$

(5$^{\text{pts}}$) **3.** Refer to (74.) on page 12 of the reading. Verify

$$\mathrm{E}[(\underset{\sim}{b} - \underset{\sim}{\beta})(\underset{\sim}{b} - \underset{\sim}{\beta})^\top] = \sigma^2(\mathbf{X}^\top\mathbf{X})^{-1}.$$

Solution: First note that

$$\begin{aligned}\underset{\sim}{b} &= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{y} \\ &= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top(\mathbf{X}\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}) \\ &= \mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}\underset{\sim}{\beta} + \mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{\varepsilon} \\ &= \underset{\sim}{\beta} + \mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{\varepsilon},\end{aligned}$$

thus,

$$\underset{\sim}{b} - \underset{\sim}{\beta} = \mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{\varepsilon}.$$

Substituting, and solving, we find

$$\begin{aligned}
\mathrm{E}[(\underset{\sim}{b}-\underset{\sim}{\beta})(\underset{\sim}{b}-\underset{\sim}{\beta})^\top] &= \mathrm{E}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{\varepsilon})(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{\varepsilon})^\top] \\
&= \mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathrm{E}[\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}^\top]\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1} \\
&= \mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\sigma^2\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1} \\
&= \sigma^2\mathbf{X}^\top\mathbf{X})^{-1}(\mathbf{X}^\top\mathbf{X})(\mathbf{X}^\top\mathbf{X})^{-1} \\
&= \sigma^2\mathbf{I}(\mathbf{X}^\top\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}^\top\mathbf{X})^{-1}.
\end{aligned}$$

(10^{pts}) **4.** In (73.) on page 12 of your notes, the claim is that $\mathrm{E}(\underset{\sim}{b}) = \underset{\sim}{\beta}$ if the model true model is $\underset{\sim}{y} = \mathbf{X}\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ and $\underset{\sim}{b} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{y}$ is the least squares estimator of $\underset{\sim}{\beta}$. Suppose, however, that the true model contains additional parameters given by vector $\underset{\sim}{\beta}_2$. Therefore, the "true" linear model should be

$$\underset{\sim}{y} = \mathbf{X}\underset{\sim}{\beta} + \mathbf{X}_2\underset{\sim}{\beta}_2 + \underset{\sim}{\varepsilon} \tag{1}$$

where $\mathrm{E}[\underset{\sim}{\varepsilon}] = \underset{\sim}{0}$ and $\mathrm{Var}[\underset{\sim}{\varepsilon}] = \sigma^2\mathbf{I}_n$. Suppose $\mathbf{X}$ is $n \times p$ and $\mathbf{X}_2$ is $n \times p_2$. Show

$$\mathrm{E}[\underset{\sim}{b}] = \underset{\sim}{\beta} + \mathbf{A}\underset{\sim}{\beta}_2,$$

where $\mathbf{A} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}_2$.

Solution: Using the true model in (1), and plugging that in for $\mathrm{E}[\underset{\sim}{y}]$, and following the same argument as in problem 2 above, we find

$$\begin{aligned}
\mathrm{E}[\underset{\sim}{b}] &= \mathrm{E}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\underset{\sim}{y}] \\
&= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathrm{E}[\underset{\sim}{y}] \\
&= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathrm{E}[\mathbf{X}\underset{\sim}{\beta} + \mathbf{X}_2\underset{\sim}{\beta}_2 + \underset{\sim}{\varepsilon}] \\
&= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top(\mathbf{X}\underset{\sim}{\beta} + \mathbf{X}_2\underset{\sim}{\beta}_2 + 0) \\
&= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}\underset{\sim}{\beta} + (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}_2\underset{\sim}{\beta}_2 \\
&= (\mathbf{X}^\top\mathbf{X})^{-1}(\mathbf{X}^\top\mathbf{X})\underset{\sim}{\beta} + (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}_2\underset{\sim}{\beta}_2 \\
&= \underset{\sim}{\beta} + (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}_2\underset{\sim}{\beta}_2.
\end{aligned}$$