Chapter 8

Polynomial Regression

8.1 Polynomial Models with One Predictor

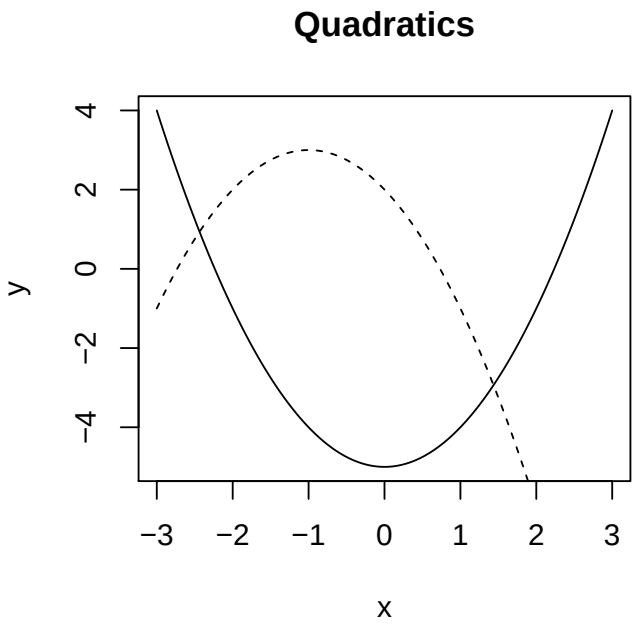
A p^{th} order polynomial model relating a dependent variable Y to a predictor X is given by

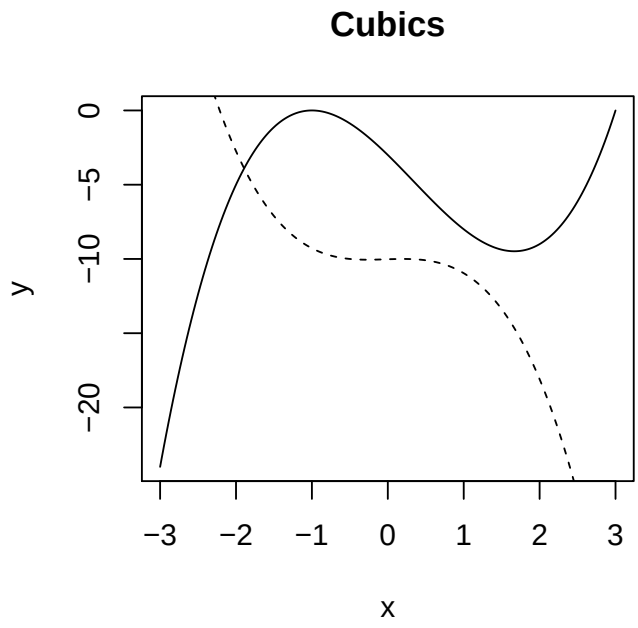
$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots + \beta_p X^p + \varepsilon.$$

This is a multiple regression model with predictors $X, X^2, \ldots, X^p$. For $p = 2$, 3, 4, and 5 we have quadratic, cubic, quartic and quintic relationships, respectively.

A second order polynomial (quadratic) allows at most one local maximum or minimum (i.e., a point where trend changes direction from increasing to decreasing, or from decreasing to increasing). A third order polynomial (cubic) allows at most two local maxima or minima. In general, a p^{th} order polynomial allows at most $p - 1$ local maxima or minima. The two panels below illustrate different quadratic and cubic relationships.

```
#### Creating polynomial plots
# R code for quadratic and cubic plots
x <- seq(-3,3,0.01);
y21 <- x^2-5;
y22 <- -(x+1)^2+3;
y31 <- (x+1)^2*(x-3);
y32 <- -(x-.2)^2*(x+.5)-10;

plot(  x, y21, type="l", main="Quadratics", ylab="y")
points(x, y22, type="l", lt=2)
plot(  x, y31, type="l", main="Cubics", ylab="y")
points(x, y32, type="l", lt=2)
```





It is important to recognize that not all, or even none, of the "turning-points" in a polynomial may be observed if the range of X is suitably restricted.

Although polynomial models allow a rich class of non-linear relationships between Y and X (by virtue of Taylor's Theorem in calculus), some caution is needed when fitting polynomials. In particular, the extreme X-values can be highly influential, numerical instabilities occur when fitting high order models, and predictions based on high order polynomial models can be woeful.

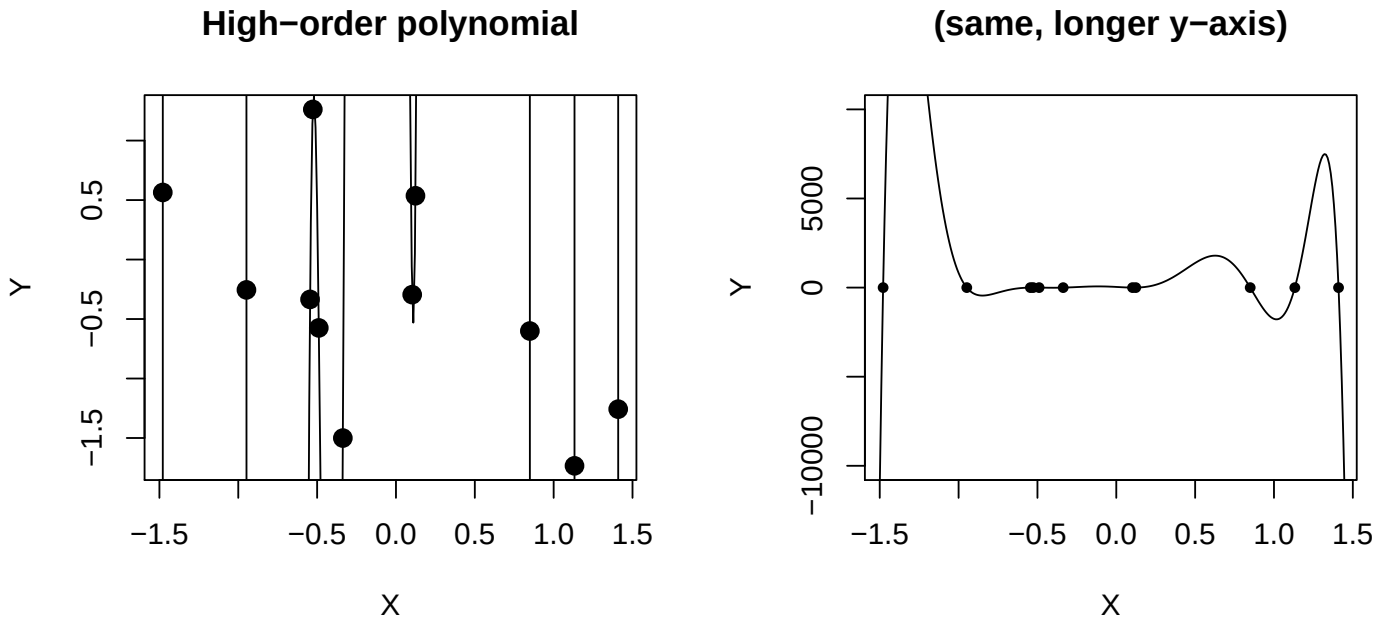
To illustrate the third concern, consider a data set (Y_i, X_i) for $i = 1, 2, \ldots, n$ where the X_is are distinct. One can show mathematically that an $(n-1)^{st}$ degree polynomial will fit the observed data exactly. However, for a high order polynomial to fit exactly, the fitted curve must oscillate wildly between data points. In the picture below, I show the 10^{th} degree polynomial that fits exactly the 11 distinct data points. Although $R^2 = 1$, I would not use the fitted model to make predictions with new data. (If possible, models should always be validated using new data.) Intuitively, a quadratic or a lower order polynomial would likely be significantly better. In essence, the 10^{th} degree polynomial is modelling the variability in the data, rather than the trend.

```
# R code for quadratic and cubic plots

X <- rnorm(11); Y <- rnorm(11);  # observed
X1  <- X^1 ;
X2  <- X^2 ;
```

```
X3  <- X^3 ;
X4  <- X^4 ;
X5  <- X^5 ;
X6  <- X^6 ;
X7  <- X^7 ;
X8  <- X^8 ;
X9  <- X^9 ;
X10 <- X^10;

fit <- lm(Y~X+X2+X3+X4+X5+X6+X7+X8+X9+X10)
fit$coefficients
```

```
##  (Intercept)            X           X2           X3           X4
##     36.70206   -461.55109   -620.55094  13030.85848  29149.14341
##           X5           X6           X7           X8           X9
## -26416.29553 -81282.20211  15955.10270  70539.53467  -3396.10960
##          X10
## -18290.46769
```

```
x <- seq(-2.5,2.5,0.01);
x1  <- x^1 ;
x2  <- x^2 ;
x3  <- x^3 ;
x4  <- x^4 ;
x5  <- x^5 ;
x6  <- x^6 ;
x7  <- x^7 ;
x8  <- x^8 ;
x9  <- x^9 ;
x10 <- x^10;

xx <- matrix(c(rep(1,length(x)),x1,x2,x3,x4,x5,x6,x7,x8,x9,x10),ncol=11)

y <- xx %*% fit$coefficients;

plot(  X, Y, main="High-order polynomial", pch=20, cex=2)
points(x, y, type="l", lt=1)
plot(  X, Y, main="(same, longer y-axis)", pch=20, cex=1, ylim=c(-10000,10000))
points(x, y, type="l", lt=1)
```



Another concern is that the importance of lower order terms (i.e., X, X^2, ..., X^{p-1}) depends on the scale in which we measure X. For example, suppose

for some chemical reaction,

$$\text{Time to reaction} = \beta_0 + \beta_1 \text{ Temp} + \beta_2 \text{ Temp}^2 + \varepsilon.$$

The significance level for the estimate of the Temp coefficient depends on whether we measure temperature in degrees Celsius or Fahrenheit.

To avoid these problems, I recommend the following:

1. Center the X data at $\bar{X}$ and fit the model

$$Y = \beta_0 + \beta_1(X - \bar{X}) + \beta_2(X - \bar{X})^2 + \cdots + \beta_p(X - \bar{X})^p + \varepsilon.$$

   This is usually important only for cubic and higher order models.
2. Restrict attention to low order models, say quartic or less. If a fourth-order polynomial does not fit, a transformation may provide a more succinct summary.
3. Pay careful attention to diagnostics.
4. Add or delete variables using the natural hierarchy among powers of X and include all lower order terms if a higher order term is needed. For example, in a forward-selection type algorithm, add terms X, X^2, ..., sequentially until no additional term is significant, but do not delete powers that were entered earlier. Similarly, with a backward-elimination type algorithm, start with the model of maximum acceptable order (for example a fourth or third order polynomial) and consider deleting terms in the order X^p, X^{p-1}, ..., until no further terms can be omitted. The **select=backward** option in the **reg** procedure **does not allow** you to invoke the hierarchy principle with backward elimination. The backward option sequentially eliminates the least significant effect in the model, regardless of what other effects are included.

8.1.1 Example: Cloud point and percent I-8

The cloud point of a liquid is a measure of the degree of crystallization in a stock, and is measured by the refractive index [1]. It has been suggested that the

[1] Draper and Smith 1966, p. 162

percent of I-8 (variable "i8") in the base stock is an excellent predictor of the cloud point using a second order (quadratic) model:

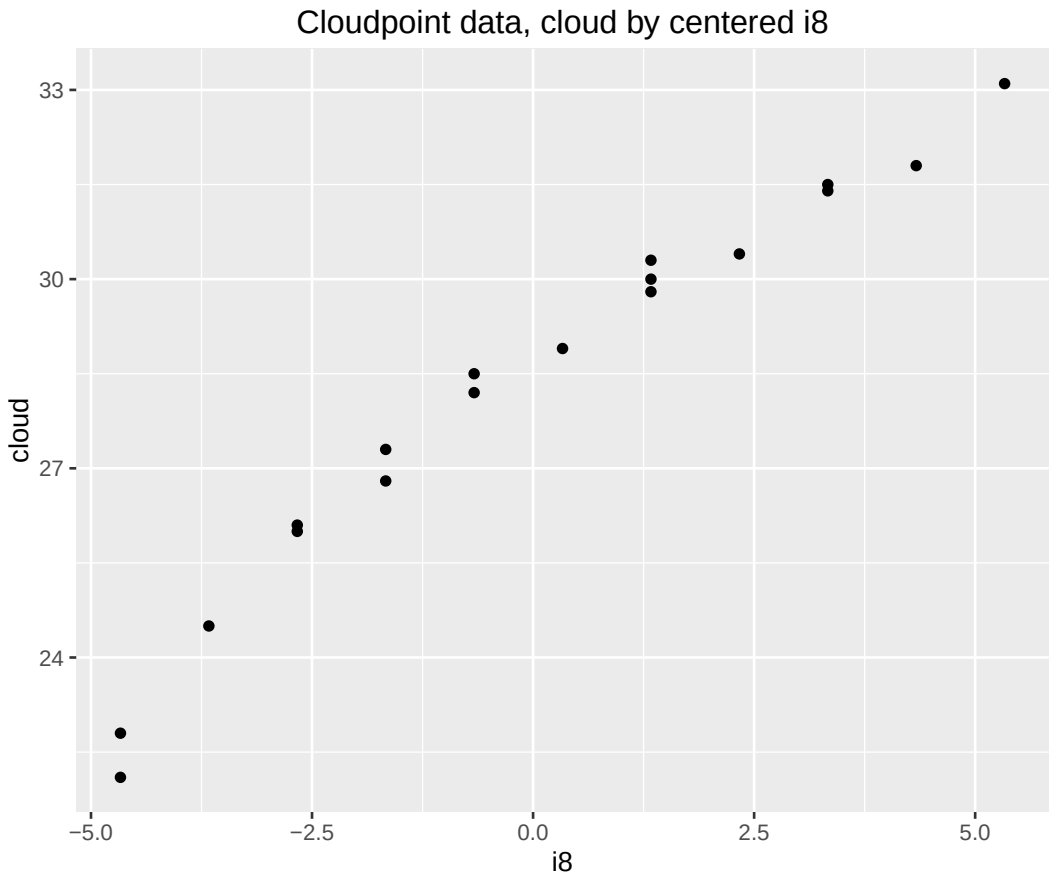
$$\text{Cloud point} = \beta_0 + \beta_1 \text{ I8} + \beta_2 \text{ I8}^2 + \varepsilon.$$

Data were collected to examine this model.

```
#### Example: Cloud point
cloudpoint <- read.table("http://statacumen.com/teach/ADA2/ADA2_notes_Ch08_cloudpoint.dat"
                         , header = TRUE)
# center i8 by subracting the mean
cloudpoint$i8 <- cloudpoint$i8 - mean(cloudpoint$i8)
```

The plot of the data suggest a departure from a linear relationship.

```
library(ggplot2)
p <- ggplot(cloudpoint, aes(x = i8, y = cloud))
p <- p + geom_point()
p <- p + labs(title="Cloudpoint data, cloud by centered i8")
print(p)
```



Fit the simple linear regression model and plot the residuals.

```
lm.c.i <- lm(cloud ~ i8, data = cloudpoint)
#library(car)
#Anova(aov(lm.c.i), type=3)
#summary(lm.c.i)
```

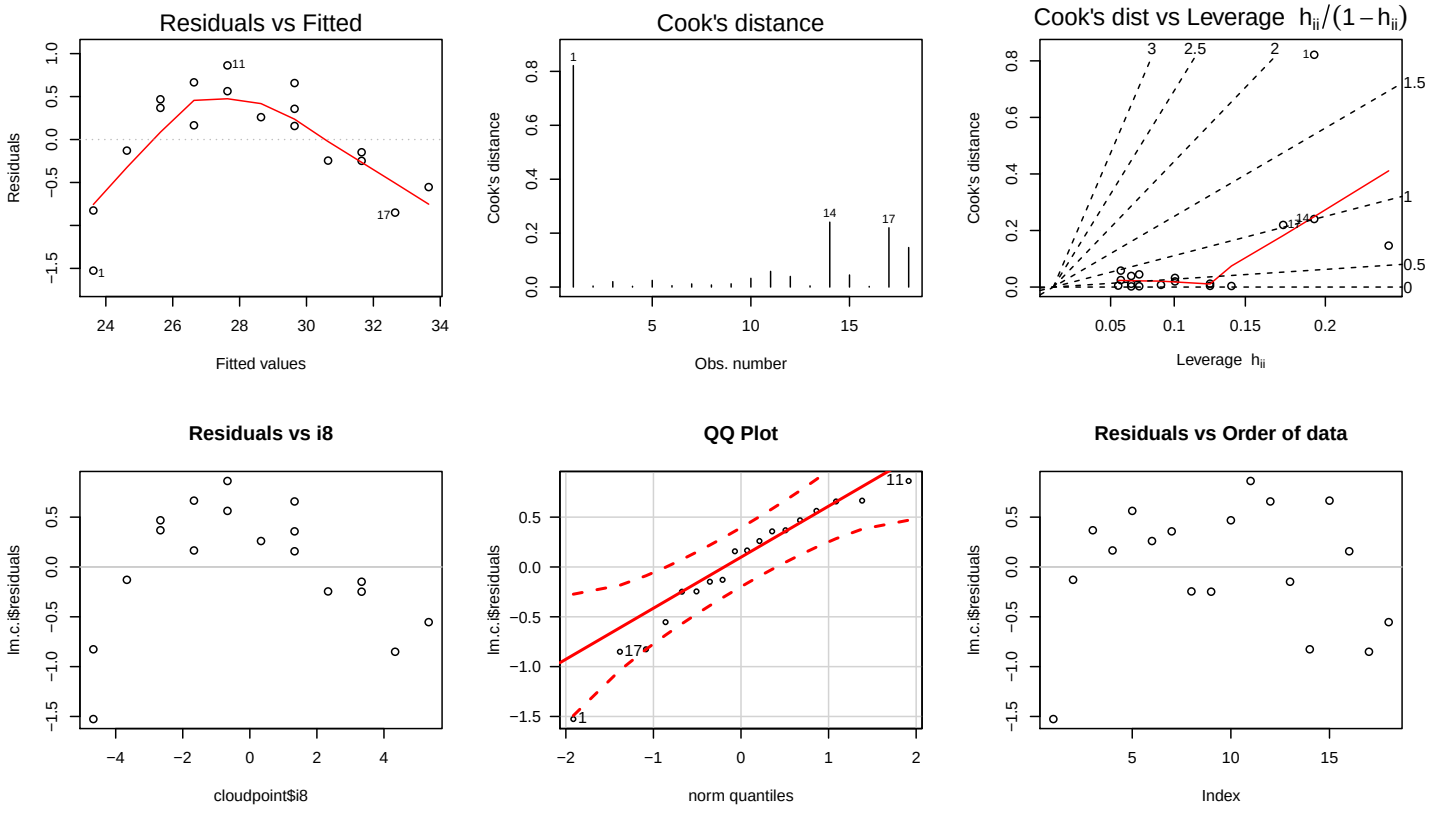
The data plot is clearly nonlinear, suggesting that a simple linear regression

model is inadequate. This is confirmed by a plot of the studentized residuals against the fitted values from a simple linear regression of Cloud point on i8. Also by the residuals against the i8 values. We do not see any local maxima or minima, so a second order model is likely to be adequate. To be sure, we will first fit a cubic model, and see whether the third order term is important.

```
# plot diagnistics
par(mfrow=c(2,3))
plot(lm.c.i, which = c(1,4,6), pch=as.character(cloudpoint$type))

plot(cloudpoint$i8, lm.c.i$residuals, main="Residuals vs i8", pch=as.character(cloudpoint$type))
  # horizontal line at zero
  abline(h = 0, col = "gray75")

# Normality of Residuals
library(car)
qqPlot(lm.c.i$residuals, las = 1, id.n = 3, main="QQ Plot", pch=as.character(cloudpoint$type))
```

```
##  1 11 17
##  1 18  2
```

```
# residuals vs order of data
plot(lm.c.i$residuals, main="Residuals vs Order of data")
  # horizontal line at zero
  abline(h = 0, col = "gray75")
```



The output below shows that the cubic term improves the fit of the quadratic model (i.e., the cubic term is important when added last to the model). The plot of the studentized residuals against the fitted values does not show any extreme abnormalities. Furthermore, no individual point is poorly fitted by the

model. Case 1 has the largest studentized residual: $r_1 = -1.997$.

```
# I() is used to create an interpreted object treated "as is"
#   so we can include quadratic and cubic terms in the formula
#   without creating separate columns in the dataset of these terms
lm.c.i3 <- lm(cloud ~ i8 + I(i8^2) + I(i8^3), data = cloudpoint)
#library(car)
#Anova(aov(lm.c.i3), type=3)
summary(lm.c.i3)
```

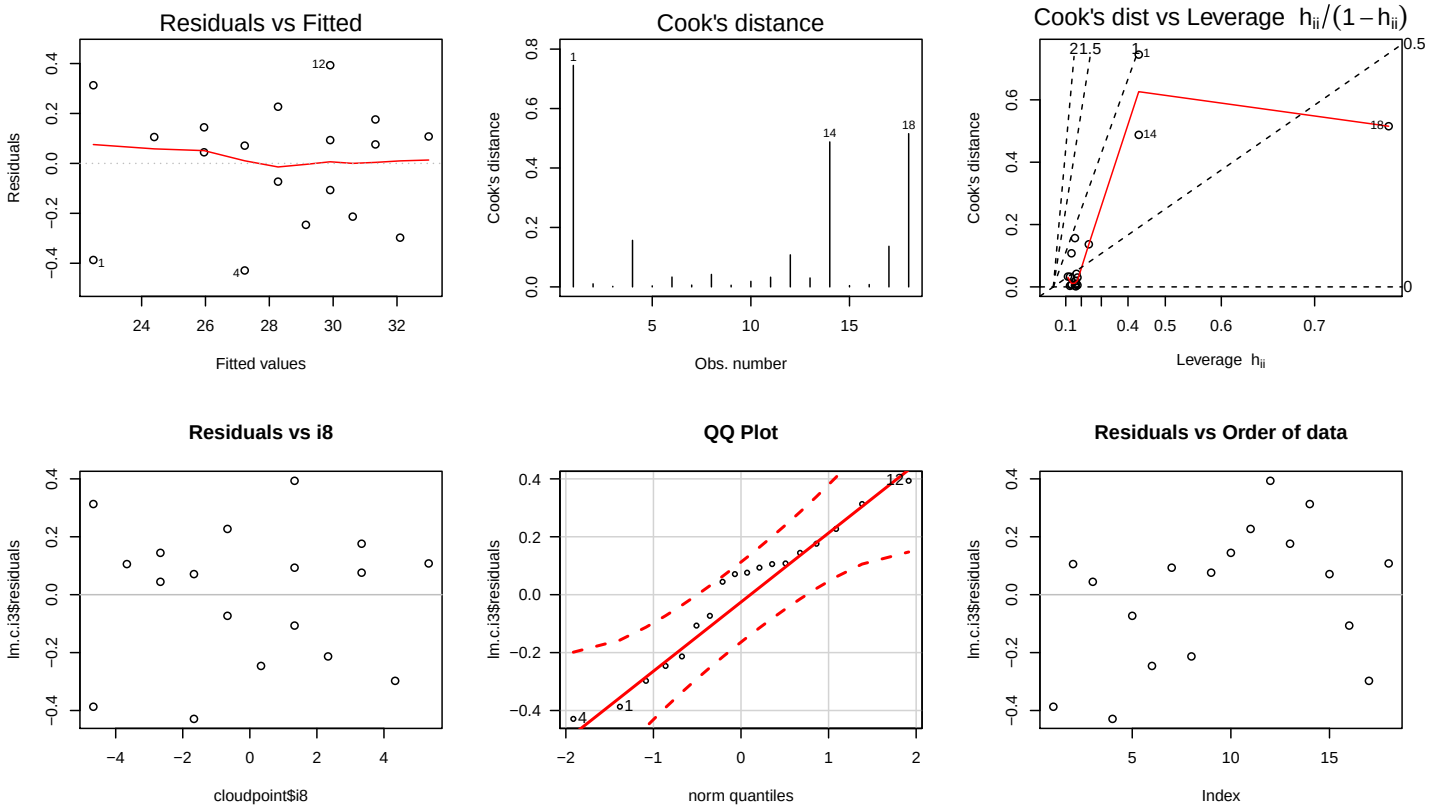
```
##
## Call:
## lm(formula = cloud ~ i8 + I(i8^2) + I(i8^3), data = cloudpoint)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.42890 -0.18658  0.07355  0.13536  0.39328
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 28.870451   0.088364 326.723  < 2e-16 ***
## i8           0.847889   0.048536  17.469 6.67e-11 ***
## I(i8^2)     -0.065998   0.007323  -9.012 3.33e-07 ***
## I(i8^3)      0.009735   0.002588   3.762   0.0021 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.2599 on 14 degrees of freedom
## Multiple R-squared:  0.9943,Adjusted R-squared:  0.9931
## F-statistic: 812.9 on 3 and 14 DF,  p-value: 6.189e-16
```

Below are plots of the data and the studentized residuals.

```
# plot diagnistics
par(mfrow=c(2,3))
plot(lm.c.i3, which = c(1,4,6), pch=as.character(cloudpoint$type))

plot(cloudpoint$i8, lm.c.i3$residuals, main="Residuals vs i8", pch=as.character(cloudpoint$type))
  # horizontal line at zero
  abline(h = 0, col = "gray75")

# Normality of Residuals
library(car)
qqPlot(lm.c.i3$residuals, las = 1, id.n = 3, main="QQ Plot", pch=as.character(cloudpoint$type))
```

```
##  4 12  1
##  1 18  2
```

```
# residuals vs order of data
plot(lm.c.i3$residuals, main="Residuals vs Order of data")
  # horizontal line at zero
  abline(h = 0, col = "gray75")
```



The first and last observations have the lowest and highest values of I8, given by 0 and 10, respectively. These cases are also the most influential points in the data set (largest Cook's D). If we delete these cases and redo the analysis we find that the cubic term is no longer important (p-value=0.55) when added after the quadratic term. One may reasonably conclude that the significance of the cubic term in the original analysis is solely due to the two extreme I8 values, and that the quadratic model appears to fit well over the smaller range of $1 \leq \text{I8} \leq 9$.

```
# remove points for minimum and maximum i8 values
cloudpoint2 <- cloudpoint[!(cloudpoint$i8 == min(cloudpoint$i8) |
                            cloudpoint$i8 == max(cloudpoint$i8)), ]
lm.c.i2 <- lm(cloud ~ i8 + I(i8^2) + I(i8^3), data = cloudpoint2)
#library(car)
#Anova(aov(lm.c.i2), type=3)
summary(lm.c.i2)
```

```
##
## Call:
## lm(formula = cloud ~ i8 + I(i8^2) + I(i8^3), data = cloudpoint2)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
```

```
## -0.36620 -0.12845  0.03737  0.14031  0.33737
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 28.857039   0.089465 322.551  < 2e-16 ***
## i8           0.904515   0.058338  15.505 8.04e-09 ***
## I(i8^2)     -0.060714   0.012692  -4.784 0.000568 ***
## I(i8^3)      0.003168   0.005166   0.613 0.552200
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.2313 on 11 degrees of freedom
## Multiple R-squared:  0.9917,Adjusted R-squared:  0.9894
## F-statistic: 436.3 on 3 and 11 DF,  p-value: 1.032e-11
```

8.2 Polynomial Models with Two Predictors

Polynomial models are sometimes fit to data collected from experiments with two or more predictors. For simplicity, consider the general quadratic model, with two predictors:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1^2 + \beta_4 X_2^2 + \beta_5 X_1 X_2 + \varepsilon.$$

The model, which can be justified as a second order approximation to a smooth trend, includes quadratic terms in X_1 and X_2 and the product or interaction of X_1 and X_2.

8.2.1 Example: Mooney viscosity

The data below give the Mooney viscosity at 100 degrees Celsius (Y) as a function of the filler level (X_1) and the naphthenic oil (X_2) level for an experiment involving filled and plasticized elastomer compounds.

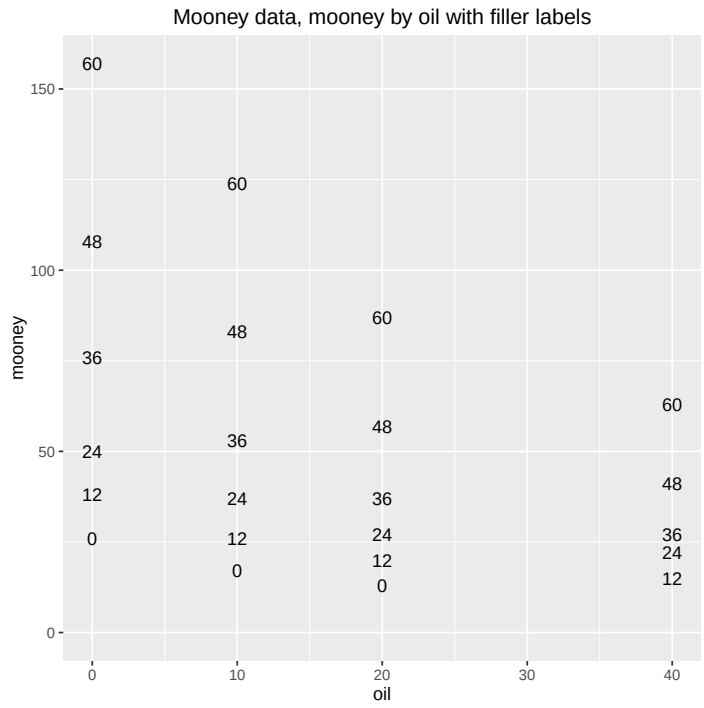
```
#### Example: Mooney viscosity
mooney <- read.table("http://statacumen.com/teach/ADA2/ADA2_notes_Ch08_mooney.dat"
                     , header = TRUE)
```

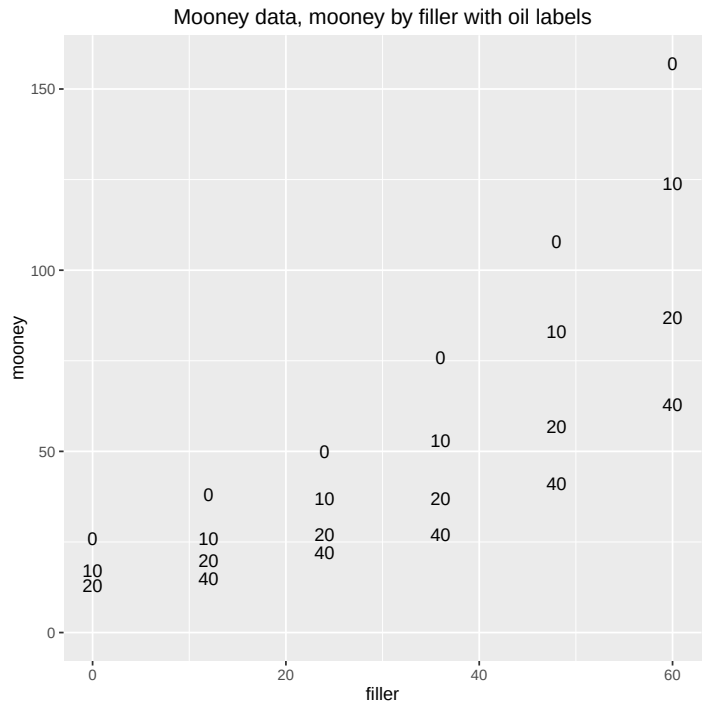
```
library(ggplot2)
p <- ggplot(mooney, aes(x = oil, y = mooney, label = filler))
p <- p + geom_text()
p <- p + scale_y_continuous(limits = c(0, max(mooney$mooney, na.rm=TRUE)))
p <- p + labs(title="Mooney data, mooney by oil with filler labels")
print(p)
```

```
## Warning:  Removed 1 rows containing missing values (geom_text).
```

```
library(ggplot2)
p <- ggplot(mooney, aes(x = filler, y = mooney, label = oil))
p <- p + geom_text()
p <- p + scale_y_continuous(limits = c(0, max(mooney$mooney, na.rm=TRUE)))
p <- p + labs(title="Mooney data, mooney by filler with oil labels")
print(p)
```

```
## Warning:  Removed 1 rows containing missing values (geom_text).
```





At each of the 4 oil levels, the relationship between the Mooney viscosity and filler level (with 6 levels) appears to be quadratic. Similarly, the relationship between the Mooney viscosity and oil level appears quadratic for each filler level (with 4 levels). This supports fitting the general quadratic model as a first step in the analysis.

The output below shows that each term is needed in the model. Although there are potentially influential points (cases 6 and 20), deleting either or both cases does not change the significance of the effects in the model (not shown).

```
# I create each term separately
lm.m.o2.f2 <- lm(mooney ~ oil + filler + I(oil^2) + I(filler^2) + I(oil * filler),
                 data = mooney)
summary(lm.m.o2.f2)
```

```
##
## Call:
```

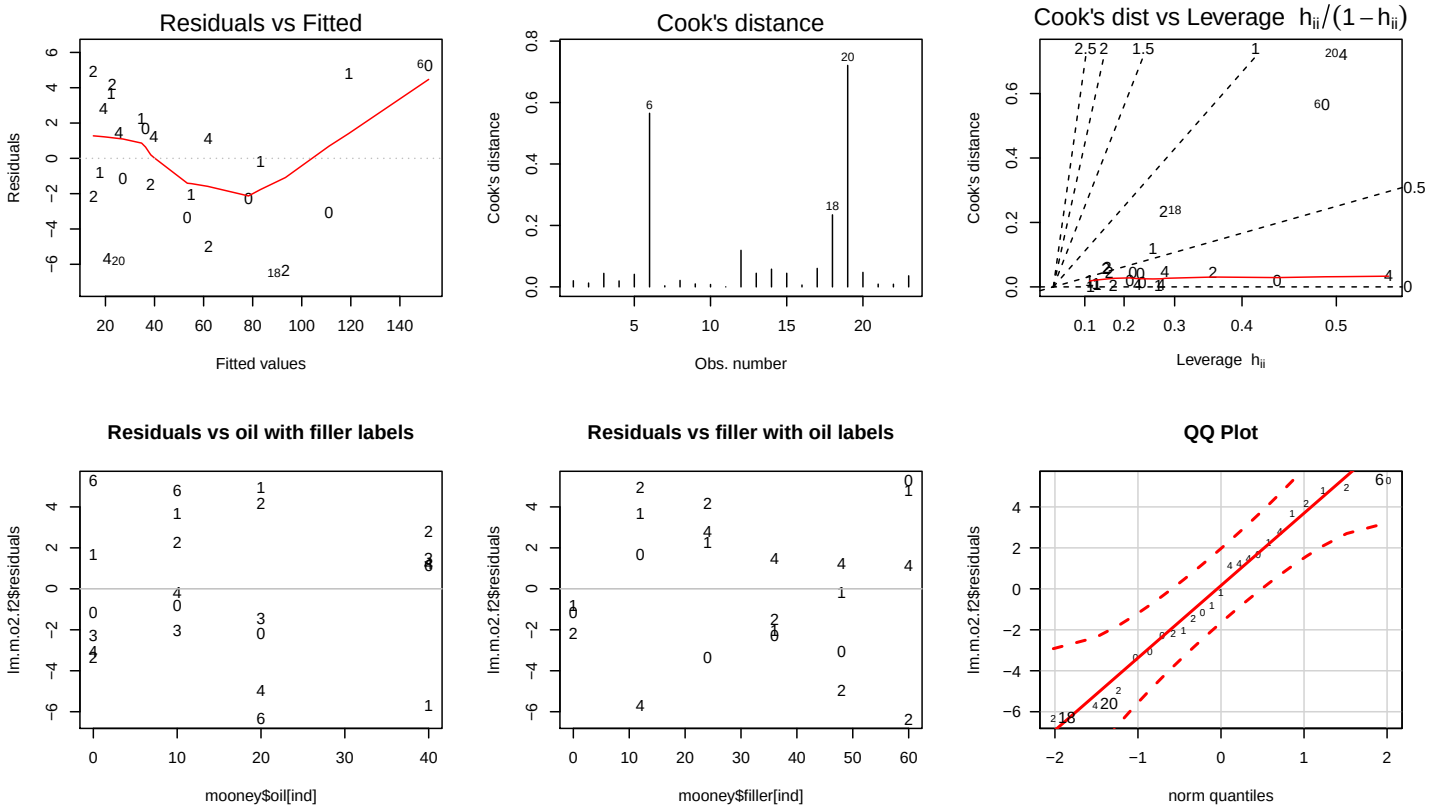
```
## lm(formula = mooney ~ oil + filler + I(oil^2) + I(filler^2) +
##     I(oil * filler), data = mooney)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.3497 -2.2231 -0.1615  2.5424  5.2749
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)     27.144582   2.616779  10.373 9.02e-09 ***
## oil             -1.271442   0.213533  -5.954 1.57e-05 ***
## filler           0.436984   0.152658   2.862   0.0108 *
## I(oil^2)         0.033611   0.004663   7.208 1.46e-06 ***
## I(filler^2)      0.027323   0.002410  11.339 2.38e-09 ***
## I(oil * filler) -0.038659   0.003187 -12.131 8.52e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.937 on 17 degrees of freedom
##   (1 observation deleted due to missingness)
## Multiple R-squared:  0.9917,Adjusted R-squared:  0.9892
## F-statistic: 405.2 on 5 and 17 DF,  p-value: < 2.2e-16
```

```
## poly() will evaluate variables and give joint polynomial values
##        which is helpful when you have many predictors
#head(mooney, 10)
#head(poly(mooney$oil, mooney$filler, degree = 2, raw = TRUE), 10)
## This model is equivalent to the one above
#lm.m.o2.f2 <- lm(mooney ~ poly(oil, filler, degree = 2, raw = TRUE), data = mooney)
#summary(lm.m.o2.f2)
```

```
# plot diagnistics
par(mfrow=c(2,3))
plot(lm.m.o2.f2, which = c(1,4,6), pch=as.character(mooney$oil))

# because of one missing value, get the indices of non-missing
ind <- as.numeric(names(lm.m.o2.f2$residuals))

plot(mooney$oil[ind], lm.m.o2.f2$residuals, main="Residuals vs oil with filler labels", pch=as.character(mooney$filler[ind]))
  # horizontal line at zero
  abline(h = 0, col = "gray75")

plot(mooney$filler[ind], lm.m.o2.f2$residuals, main="Residuals vs filler with oil labels", pch=as.character(mooney$oil[ind]))
  # horizontal line at zero
  abline(h = 0, col = "gray75")

# Normality of Residuals
library(car)
qqPlot(lm.m.o2.f2$residuals, las = 1, id.n = 3, main="QQ Plot", pch=as.character(mooney$oil[ind]))
```

```
## 18 20  6
##  1  2 23
```

```
## residuals vs order of data
#plot(lm.m.o2.f2$residuals, main="Residuals vs Order of data")
#  # horizontal line at zero
#  abline(h = 0, col = "gray75")
```



8.2.2 Example: Mooney viscosity on log scale

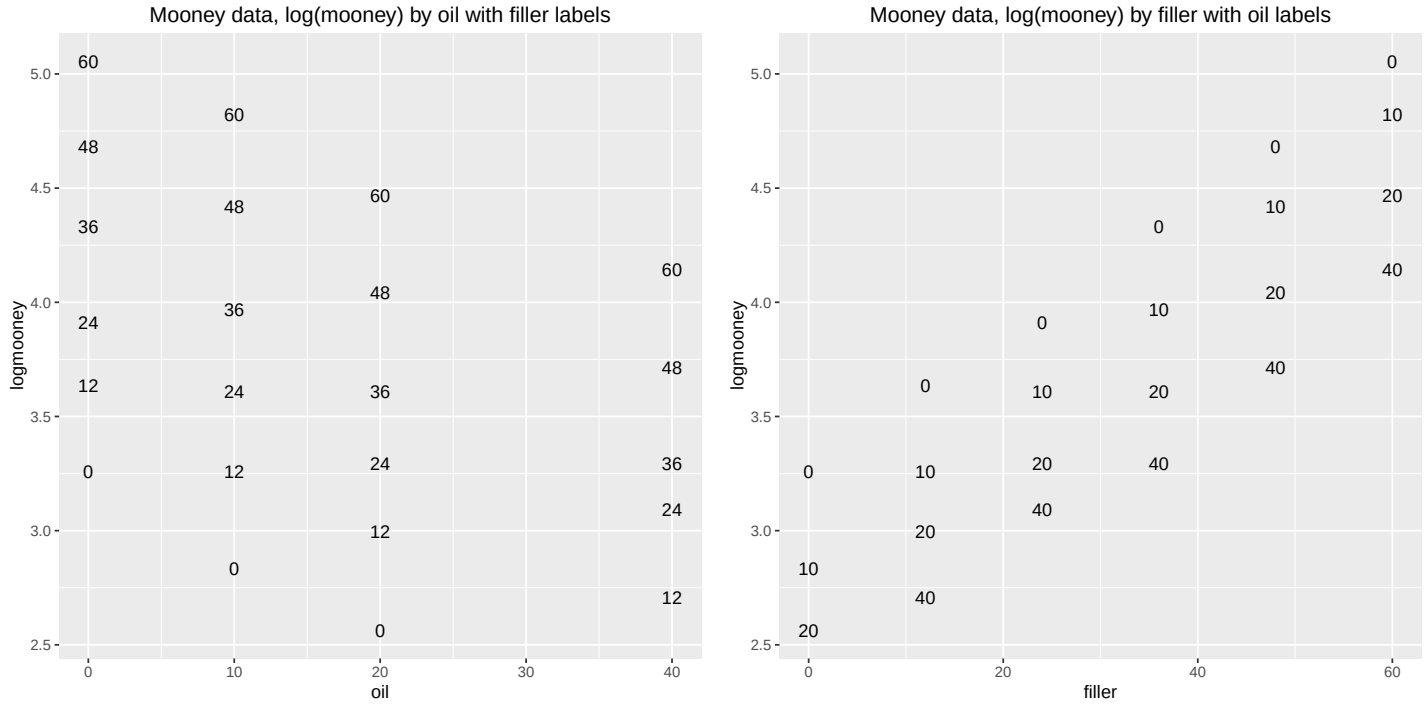
As noted earlier, transformations can often be used instead of polynomials. For example, the original data plots suggest transforming the Moody viscosity to a log scale. If we make this transformation and replot the data, we see that the log Mooney viscosity is roughly linearly related to the filler level at each oil level, but is a quadratic function of oil at each filler level. The plots of the transformed data suggest that a simpler model will be appropriate.

```
# log transform the response
mooney$logmooney <- log(mooney$mooney)

library(ggplot2)
p <- ggplot(mooney, aes(x = oil, y = logmooney, label = filler))
p <- p + geom_text()
#p <- p + scale_y_continuous(limits = c(0, max(mooney$logmooney, na.rm=TRUE)))
p <- p + labs(title="Mooney data, log(mooney) by oil with filler labels")
print(p)
```

```
## Warning:  Removed 1 rows containing missing values (geom_text).
```

```
library(ggplot2)
p <- ggplot(mooney, aes(x = filler, y = logmooney, label = oil))
p <- p + geom_text()
#p <- p + scale_y_continuous(limits = c(0, max(mooney$logmooney, na.rm=TRUE)))
p <- p + labs(title="Mooney data, log(mooney) by filler with oil labels")
print(p)
```

```
## Warning:  Removed 1 rows containing missing values (geom_text).
```



To see that a simpler model is appropriate, we fit the full quadratic model. The interaction term can be omitted here, without much loss of predictive ability (R-squared is similar). The p-value for the interaction term in the quadratic model is 0.34.

```
# I create each term separately
lm.lm.o2.f2 <- lm(logmooney ~ oil + filler + I(oil^2) + I(filler^2) + I(oil * filler),
                  data = mooney)
summary(lm.lm.o2.f2)
```

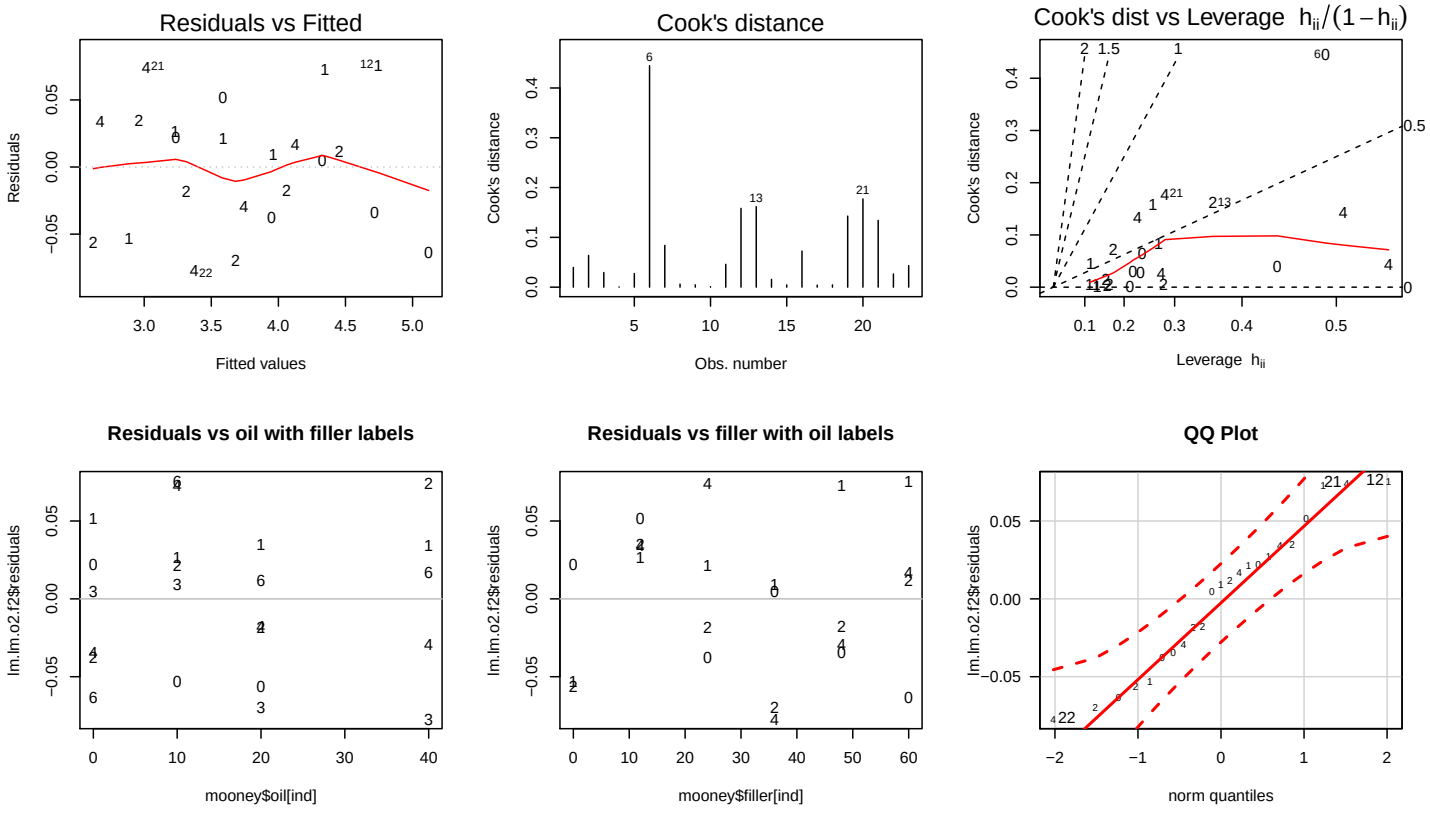
```
##
## Call:
## lm(formula = logmooney ~ oil + filler + I(oil^2) + I(filler^2) +
##     I(oil * filler), data = mooney)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.077261 -0.035795  0.009193  0.030641  0.075640
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)      3.236e+00  3.557e-02  90.970  < 2e-16 ***
## oil             -3.921e-02  2.903e-03 -13.507 1.61e-10 ***
## filler           2.860e-02  2.075e-03  13.781 1.18e-10 ***
## I(oil^2)         4.227e-04  6.339e-05   6.668 3.96e-06 ***
## I(filler^2)      4.657e-05  3.276e-05   1.421    0.173
## I(oil * filler) -4.231e-05  4.332e-05  -0.977    0.342
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
##
## Residual standard error: 0.05352 on 17 degrees of freedom
##   (1 observation deleted due to missingness)
## Multiple R-squared:  0.9954,Adjusted R-squared:  0.9941
## F-statistic:   737 on 5 and 17 DF,  p-value: < 2.2e-16
```

```
# plot diagnistics
par(mfrow=c(2,3))
plot(lm.lm.o2.f2, which = c(1,4,6), pch=as.character(mooney$oil))

# because of one missing value, get the indices of non-missing
ind <- as.numeric(names(lm.lm.o2.f2$residuals))

plot(mooney$oil[ind], lm.lm.o2.f2$residuals, main="Residuals vs oil with filler labels", pch=as.character(mooney$filler[ind]))
  # horizontal line at zero
  abline(h = 0, col = "gray75")

plot(mooney$filler[ind], lm.lm.o2.f2$residuals, main="Residuals vs filler with oil labels", pch=as.character(mooney$oil[ind]))
  # horizontal line at zero
  abline(h = 0, col = "gray75")

# Normality of Residuals
library(car)
qqPlot(lm.lm.o2.f2$residuals, las = 1, id.n = 3, main="QQ Plot", pch=as.character(mooney$oil[ind]))
```

```
## 22 12 21
##  1 23 22
```

```
## residuals vs order of data
#plot(lm.lm.o2.f2$residuals, main="Residuals vs Order of data")
#  # horizontal line at zero
#  abline(h = 0, col = "gray75")
```



After omitting the interaction term, the quadratic effect in filler is not needed in the model (output not given). Once these two effects are removed, each of

the remaining effects is significant.

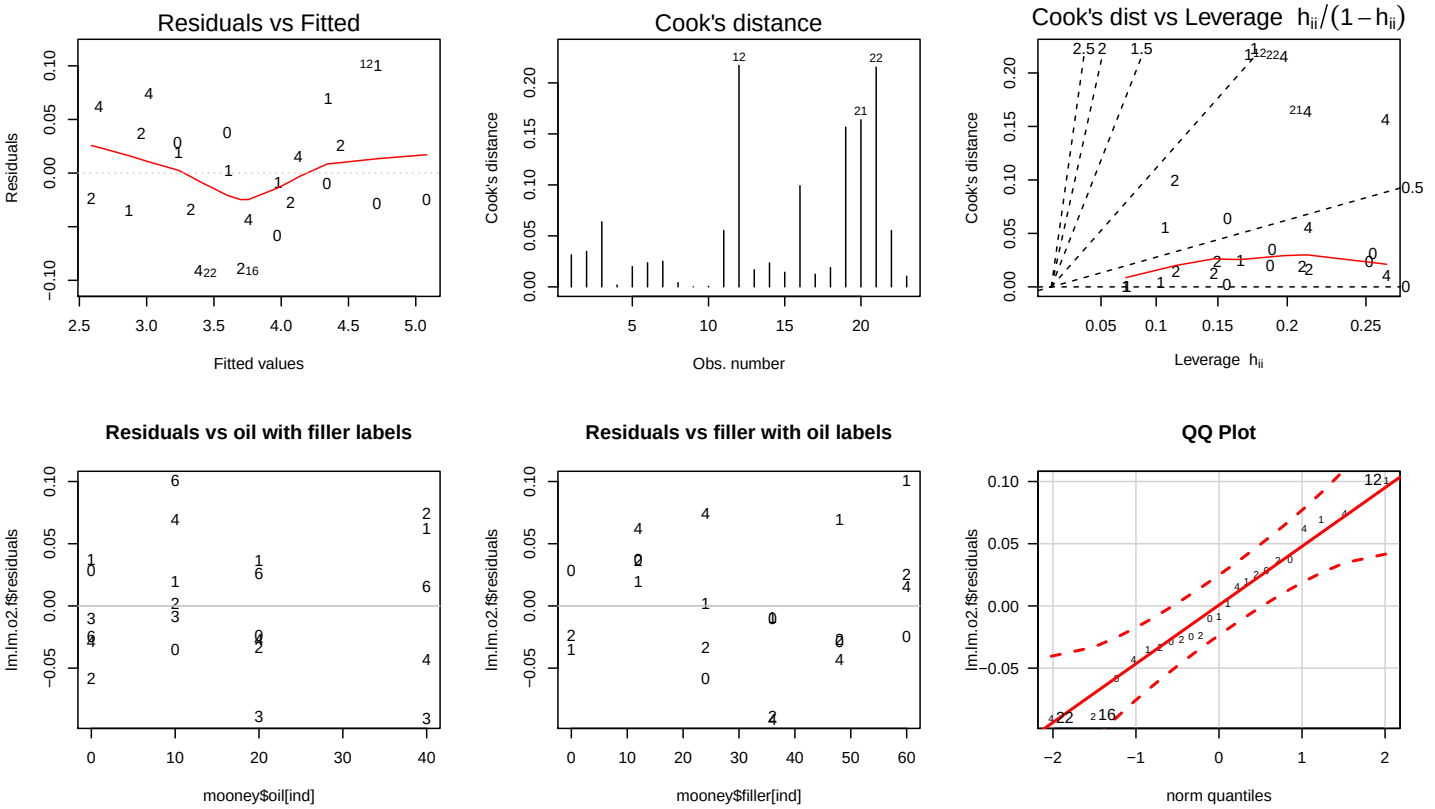
```
# I create each term separately
lm.lm.o2.f <- lm(logmooney ~ oil + filler + I(oil^2),
                    data = mooney)
summary(lm.lm.o2.f)
```

```
##
## Call:
## lm(formula = logmooney ~ oil + filler + I(oil^2), data = mooney)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.090796 -0.031113 -0.008831  0.032533  0.100587
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  3.230e+00  2.734e-02 118.139  < 2e-16 ***
## oil         -4.024e-02  2.702e-03 -14.890 6.26e-12 ***
## filler       3.086e-02  5.716e-04  53.986  < 2e-16 ***
## I(oil^2)     4.097e-04  6.356e-05   6.446 3.53e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.05423 on 19 degrees of freedom
##   (1 observation deleted due to missingness)
## Multiple R-squared:  0.9947,Adjusted R-squared:  0.9939
## F-statistic:  1195 on 3 and 19 DF,  p-value: < 2.2e-16
```

```
# plot diagnistics
par(mfrow=c(2,3))
plot(lm.lm.o2.f, which = c(1,4,6), pch=as.character(mooney$oil))

# because of one missing value, get the indices of non-missing
ind <- as.numeric(names(lm.lm.o2.f$residuals))

plot(mooney$oil[ind], lm.lm.o2.f$residuals, main="Residuals vs oil with filler labels", pch=as.character(mooney$filler[ind]))
  # horizontal line at zero
  abline(h = 0, col = "gray75")

plot(mooney$filler[ind], lm.lm.o2.f$residuals, main="Residuals vs filler with oil labels", pch=as.character(mooney$oil[ind]))
  # horizontal line at zero
  abline(h = 0, col = "gray75")

# Normality of Residuals
library(car)
qqPlot(lm.lm.o2.f$residuals, las = 1, id.n = 3, main="QQ Plot", pch=as.character(mooney$oil[ind]))
```

```
## 12 22 16
## 23  1  2
```

```
## residuals vs order of data
#plot(lm.lm.o2.f$residuals, main="Residuals vs Order of data")
#  # horizontal line at zero
#  abline(h = 0, col = "gray75")
```



The model does not appear to have inadequacies. Assuming no difficulties, an important effect of the transformation is that the resulting model is simpler than the model selected on the original scale. This is satisfying to me, but you may not agree. Note that the selected model, with linear effects due to the oil and filler levels and a quadratic effect due to the oil level, agrees with our visual assessment of the data. Assuming no inadequacies, the predicted log Moody viscosity is given by

$$\widehat{\log(\text{Moody viscosity})} = 3.2297 - 0.0402\ \text{Oil} + 0.0004\ \text{Oil}^2 + 0.0309\ \text{Filler}.$$

Quadratic models with two or more predictors are often used in industrial experiments to estimate the optimal combination of predictor values to maximize or minimize the response, over the range of predictor variable values where the model is reasonable. (This strategy is called "response surface methodology".) For example, we might wish to know what combination of oil level between 0 and 40 and filler level between 0 and 60 provides the lowest predicted Mooney viscosity (on the original or log scale). We can visually approximate the mini-

mizer using the data plots, but one can do a more careful job of analysis using standard tools from calculus.