THE BORROMEAN RINGS

by

Erik B. Erhardt

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Approved by ______________________________
Chairperson of Supervisory Committee

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to Offer Degree______________________________

Date ______________________________

Franklin Pierce College

Abstract

THE BORROMEAN RINGS

by Erik B. Erhardt

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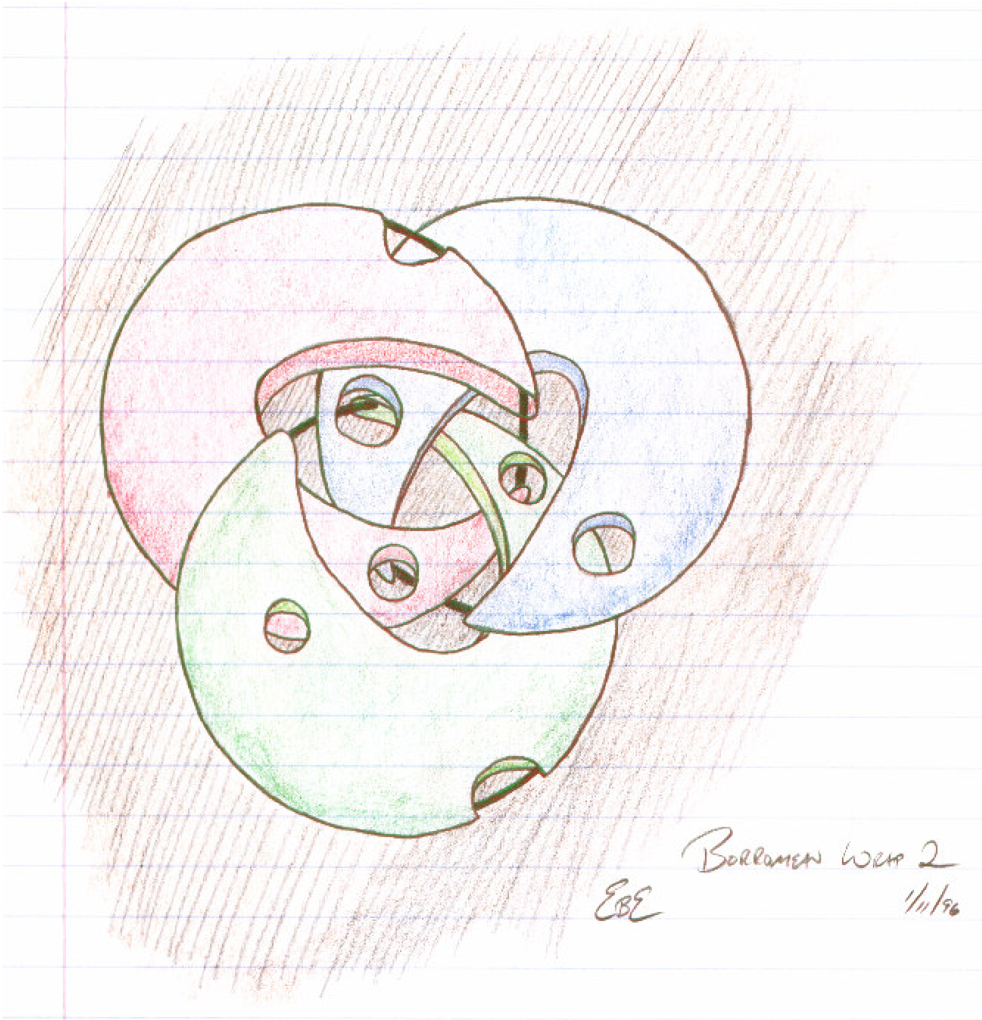
Department of Mathematics

This thesis describes the Borromean Rings, a linkage of three rings which are pairwise disjoint, but together all three are linked. Therefore, removal of any one ring causes the link to fall apart. A brief history of the Borromean Rings is given. The relevant theorems and ideas from knot theory are discussed and developed before introducing the Rings in a formal setting. Theorems specific to the Rings are given with their proofs. The Brunnian extension is discussed, which generalizes the Borromean Rings' properties to *n* rings. Notes on applications and other points are given.


Borromean Wrap 2
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ACKNOWLEDGMENTS

There are many people whose ideas, advice and encouragement helped to make this thesis possible. This list is not an exhaustive one. Thank you Mom and Dad for allowing me to keep to myself and daydream, for that is where the magic is. Thank you Carl Brezovec for supporting me in whatever I have chosen to do; it has been with your guidance that I have managed these last years so well. Thank you John Valente for providing room both in your classroom and in your day for me to come to you with the burning questions of a young mathematician. Thank you Frank Behrens for introducing me to Wagner's Ring, and thereby giving me a form of the rich Norse mythology. Thanks to my brother Mark for supporting me through even the most ridiculous of ideas. Thank you Mary Ledoux, Meredith Martin and Melissa Stearns (FPC's three critical Ms) for your patience with and generosity towards me; it is in your care that I managed to grow so well as an academic. Thank you Tony Krier for your good humor, friendship and help in finding literary leaves from mathematics. Thank you Orion, my great warrior friend, for listening and showing your southern presence with hanging scabbard from belt starred three.

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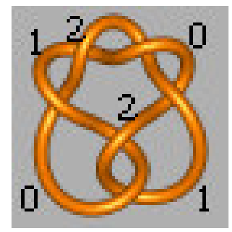
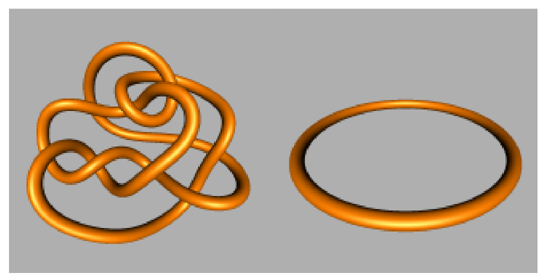
Colorable. A knot of a link K is said to be colorable mod n ($n \geq 3$) if K has a diagram D in which it is possible to assign an integer to each arc of D which does not contain an undercrossing of D such that:

1. At each crossing we have $a+c=2b(\text{mod } n)$ where b is the integer assigned to the overcrossing and a and c are the integers assigned to the other two arcs and

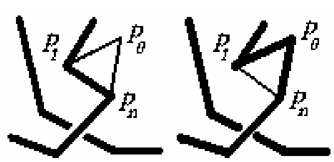
2. At least 2 distinct integers mod n are used in the diagram.

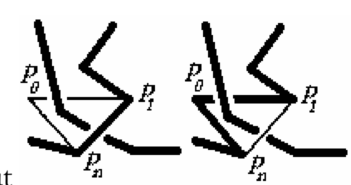


Equivalence. Knots K and J are called equivalent if there is a sequence of knots $K=K_0,K_1,\ldots,K_n=J$, with each K_{i+1} an elementary deformation of K_i, for $i \in N$.



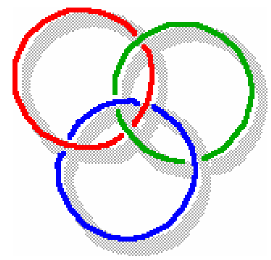
Equivalent via elementary deformations. A knot J is called an elementary deformation of the knot K if one of the two knots is determined by a sequence of points $(p_1,p_2,\ldots,p_n)$ and the other is determined by the sequence $(p_0,p_1,p_2,\ldots,p_n)$, where (1) p_0 is a point which is not collinear with p_1 and p_n, and (2) the triangle spanned by (p_0,p_1,p_n) intersects the knot determined by $(p_1,p_2,\ldots,p_n)$ only in the segment $[p_1,p_n]$.

So 

is allowed, but 

is not.

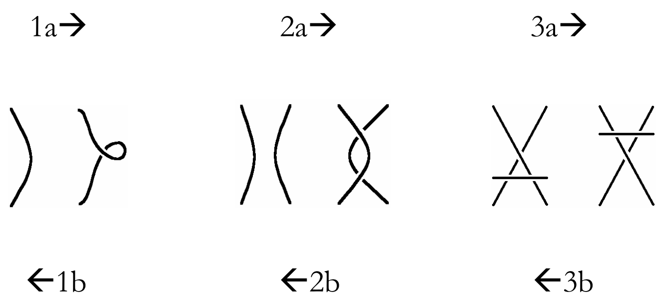
Knot. A knot is a simple closed polygonal curve in $\mathfrak{R}^3$.



Knot projection. A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.

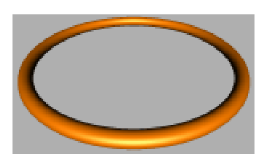
Link. A link is the finite union of disjoint knots.

Reidemeister moves. There are six simple operations which can be performed on a knot diagram without altering the corresponding knot called Reidemeister moves.



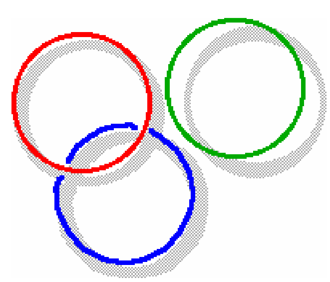
Smallest set of vertices. If the ordered set $(p_1,p_2,\ldots,p_n)$ defines a knot, and no proper ordered subset defines the same knot, the elements of the set $\{p_i\}$ are called vertices of the knot.

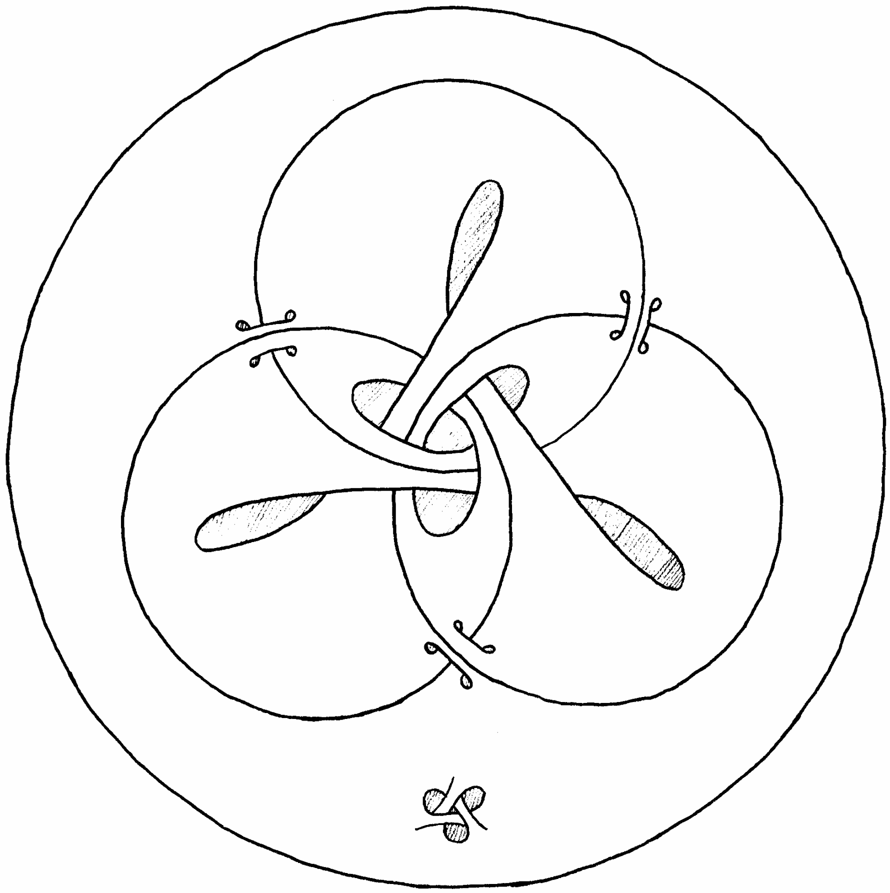
Splittable. A link L is splittable if there exists a smooth (or polyhedral) 3-ball B, an ordering of the components of the link $K_1,K_2,\ldots,K_m$ and an integer $0<k<m$ such that $K_j\subset B$ for $j\leq k$ and $K_i\subset S^3-B$ for $i>k$.



Unknot. The unknot is topologically equivalent to a circle in a plane.

Unlink. The unlink is the union of unknots all lying in a plane.





Personal Note

My musings on the Borromean rings began when I read Ian Stewart's column if the January 1994 Scientific American. These three circles are pairwise unlinked but are together linked. One has freedom only by breaking itself or when either of the others break, for breaking any one will cause the whole to disseminate. I am reminded of Socrates who thought that if a person has one virtue then he must have them all (and the contrapositive, if he does not have them all, then he has none). For if a person has any goodness, he must possess it undividedly. I am reminded of language which does not exist without the trio of grammar, lexicon and driver (or assembler). I think of other threes with the same property: Two people and love, the three graces, the three gray sisters, the three Gorgan sisters (Medusa being one), the three bears (the best being the neuter), the three phases of the moon, the three stages of exposition (thesis, antithesis and synthesis), the three kingdoms of nature, the three cardinal colors, the possible fates of the universe (heat's death or the big crunch, held together by the question of gravity), the three norns, and the three roots of Yggdrasil (the world ash tree). The Borromean trinity has many symbolic significances, all to be discussed over tea and Bach in good company.

This linkage, the more I lived with knowing of it, took increasingly long chunks of my time. I drew it in corners of notebook pages, on chalk boards of empty classrooms, or envisioned it in the empty space before me, bending and rotating.

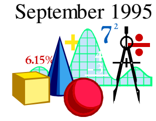
The skill I lacked in social graces I made up for in my use of a roll of string and a lighter. I spent hours cutting lengths of string and mending the molten ends together to create the Borromean Rings. It wasn't long before I thought of the possibility of four. But each time I tried to construct the linkage with four rings, the braiding pattern that worked for three would, with four, just create the Borromean rings with one strand duplicated.

I followed my instinct, maybe it has to be prime? But now two strands would be doubled when braiding five. I stopped playing with string and began scribbling knot diagrams, but I could not get it to work with more than three. (I see now that I was too stuck on the braiding idea.) Next I thought it must have been a question of dimension, questions of which I was never sure. So over a period of two months in the fall semester of my junior year spent in London, I took walks, not through the Tate Gallery, but through Erik's Mental Gallery (a colorful place, if not of the working images, then of the language of failed ones). Here I attempted to find manifolds for these linkages. I was trying to find whether what I called the Borromean formations had a countability. I wanted to find how many distinct formations existed in each dimension.

Here again I was unsure because my knowledge of surfaces and hyperbolic geometry is minimal. I believed to have exhausted the cases for dimensions two and three but when I was reaching for four I, quite naturally, came up short, for all my ideas about these rings I was, until this point, able to envision. At this point I gave up concluding that the answer might be the countability numbered 2 in Figure 1.

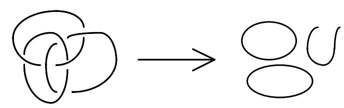
Erik B. Erhardt

September 1995



<u>Borromean Extension</u>

Goal: To find a general countability of Borromean formations, β_n , and a method for determining the manifold type and ring organization on the manifold to conform with the properties defined below.



1. The Borromean rings are three rings which are pairwise unlinked, yet all three together are linked. Therefore cutting one ring results in all of the rings becoming unlinked. These three properties are important:
   - pairwise unlinked
   - all rings together are linked
   - cutting any ring will result in unlinking them all

2. When using β_n it refers to the above Borromean properties and the following:
   - n = number of rings
   - n = dimension borromean formation resides in
   - m = dimension of manifold (where m = 1, 2, 3, ... , n)
   - r = dimension of ring (where r = 1, 2, 3, ... , m)
   - n! = number of manifolds

I hope to prove the last of the above statements, that the number of distinct manifolds in which a borromean formation can exist is equal to the factorial of the dimension the manifold resides in. The goal is countability.

Figure 1 First Attempt

I know now that I was on the wrong path the whole way. This I don't mind terribly since I recognize it as the learning process exactly. However it did take a year to discover my error though I thought I knew for nearly that whole time why I was wrong. It has to do with rising into the fourth dimension. More on this when I prove Theorem 6.

I discovered during the winter recess of 1996-97 that what I was looking for, an extension for *n* rings, had been done one-hundred years ago. Though I have yet to find the original text, German mathematician Brunn described a general crossing pattern that gives any number of rings the Borromean property. I knew heart break when I discovered that my New World had been discovered a century before. Dems de breaks.

Enjoy.

A BRIEF HISTORY OF THE BORROMEAN RINGS

The linkage gets its name from the Borromeas family of Italy who used the figure as a heraldic device in the Renaissance. One often comes across it in Italy as the coat of arms of the BORROMAEI. The Borromean Rings are carved in the stone of their castle on an island in Lago Maggiore in Northern Italy. The three rings are interlocked in such a way that if any of them were removed, the other two would also fall apart.

There is another interesting historical context in which the rings arise. The diagram was found in picture-stones on Gotland, an island in the Baltic sea off the southeast coast of Sweden. These are dated around the ninth century and are thought to tell tales from the Norse myths. To the Norse people of Scandinavia, a drawing of the Borromean rings using triangles is known as "Odin's triangle" or the "Walknot" (or "valknut" — the knot of the slain). The Walknot is the sign of Wodan and was used both in a triple and a unicursal form, though only by those given to Wodan. The symbol was also carved on the bedposts used in their burials at sea.

Figure 2 Odin and Two Ravens

Odin's sacred bird is the raven, and his principal weapon — in addition to his powerful runes, or magical spells — is the spear. He is depicted as tall, bearded, and one-eyed, having exchanged his other eye for wisdom. This is shown in Figure 2. In pre-Christian Scandinavia the Odin cult was apparently characterized by human sacrifice, which was usually accomplished by hanging the victim from a tree. The German form of his name is Woden, or Wotan; the name Wednesday is derived from Woden's day.

Figure 3 shows a picture of a 9th century Scandinavian Runes Stone, showing the Star of Wotan. This stone is similar to the memorial stone from Lärbro Stora Hammars in Gotland which now resides at the Statens Historiska Museum in Stockholm, Sweden.

Figure 3 Scandinavian Runes Stone

Other Walknots are shown in Figure 4 and Figure 5.

Figure 4
Walknot

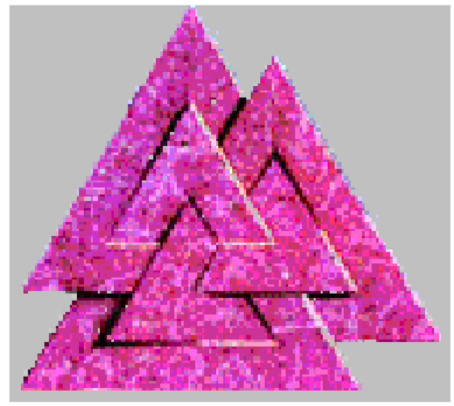


Figure 5
Walknot

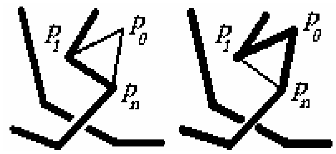
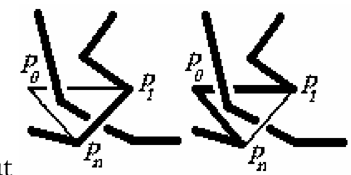
Chapter 2

ESSENTIALS FROM KNOT THEORY

A knot is a simple closed polygonal curve in $\Re^3$. A knot is often drawn smoothly for aesthetic reasons, but all figures can be drawn polygonal instead. The smallest set of vertices defining a knot is the ordered set $(p_1,p_2,\ldots,p_n)$ where no proper ordered subset defines the same knot. The elements of the set $\{p_i\}$ are called vertices of the knot. By this, if three points are collinear, eliminating the middle one does not change the underlying knot. A link is the finite union of disjoint knots. Therefore a knot is a link with one component. The unlink is the union of unknots all lying in a plane.

A knot diagram consists of a collection of arcs in the plane. These arcs are called either edges or arcs of the diagram. The points in the diagram which correspond to double points in the projection are called crossing points, or just crossings. Above the crossing point are two segments on the knot; one is called an overpass or overcrossing, the other the underpass or undercrossing. Notice that the number of arcs is the same as the number of crossings. Verify this for yourself in Figure 6.

A knot J is called an elementary deformation of the knot K if one of the two knots is determined by a sequence of points $(p_1,p_2,\ldots,p_n)$ and the other is determined by the sequence $(p_0,p_1,p_2,\ldots,p_n)$, where (1) p_0 is a point which is not collinear with p_1 and p_n, and (2) the triangle spanned by (p_0,p_1,p_n) intersects the knot determined by $(p_1,p_2,\ldots,p_n)$ only in the segment $[p_1,p_n]$.

So  is allowed, but  is not.

Knots K and J are called equivalent if there is a sequence of knots $K=K_0,K_1,\ldots,K_n=J$, with each K_{i+1} an elementary deformation of K_i, for $i\in N$. A knot projection is essentially a two dimensional shadow cast by a knot with the crossing information preserved. A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot. Knots determine equivalence classes of knots. Figure 6 shows the first eight equivalence classes.

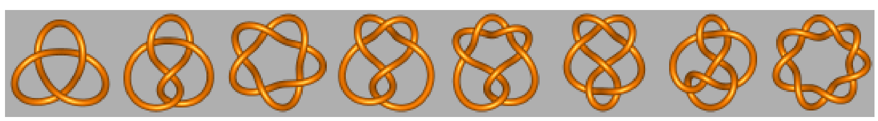


Figure 6 The First Eight Knot Equivalence Classes

Theorem 1

If knots K and J have regular projections and identical diagrams, then they are equivalent.

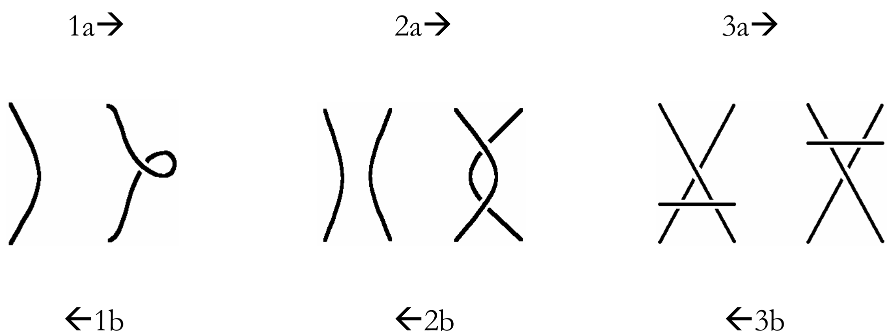
Proof: Adapted from Livinston. First arrange that K is determined by an ordered sequence $(p_1,p_2,\ldots,p_n)$ and J is determined by the sequence $(q_1,q_2,\ldots,q_n)$ with the projection of p_i and q_i the same for all i. This may require introducing extra points in the defining sequences for both knots.

Next, perform a sequence of elementary deformations that replace each p_i with a q_i in the defining sequence for K. These moves are first applied to all vertices which do not bound intervals whose projections contain crossing points. Finally each crossing point can be handled in the same way. ❑

Note that the two trefoil knots in Figure 7 are not equivalent.

Figure 7 Two Non-equivalent Knots

There are six simple operations called Reidemeister moves which can be performed on a knot diagram without altering the corresponding knot. They are:



Theorem 2

Two knots or links K_1 and K_2 are equivalent if there is an orientation preserving homeomorphism $h:\mathfrak{R}^3 \rightarrow \mathfrak{R}^3$ such that $h(K_1)=K_2$. Two knots or link diagrams D_1 and D_2 are equivalent if D_1 can be obtained from D_2 by:

1. deformations of the plane which do not alter the crossing information at each crossing point; and

2. the three Reidemeister moves and their inverses.

Proof: Adapted from Alexander and Briggs. The full proof is a detailed argument keeping track of a number of cases, but the main ideas are fairly simple.

Suppose that K and J represent equivalent knots, and that both have regular projections. Then K and J are related by a sequence of knots, each obtained from the next by an elementary deformation. A small rotation will assure that each knot in the sequence has a regular projection, and thus the proof is reduced to the case of knots related by a single elementary deformation.

Again after performing a slight rotation, it can be assured that the triangle along which the elementary deformation was performed projects to a triangle in the plane. That planar triangle might contain many crossings of the knot diagram. However, it can be divided into many small triangles, each of which contains at most one crossing.

This division can be used to describe the single elementary deformation in a sequence of many small elementary deformations; the effect of each on the diagram is quite simple. The proof is completed by checking that only Reidemeister moves have been applied. ❑

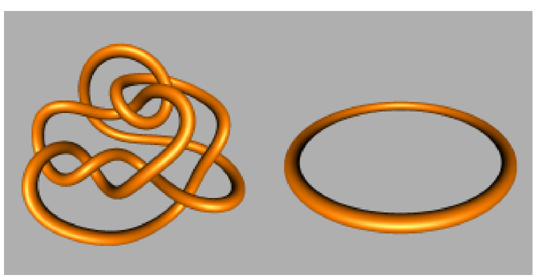


Figure 8 Two Unknots

The reader should show that the diagrams in Figure 8 are equivalent using elementary deformations and Reidemeister moves.

A knot of a link K is said to be colorable mod n ($n \geq 3$) if K has a diagram D in which it is possible to assign an integer to each arc of D which does not contain an undercrossing of D such that:

1. at each crossing we have $a+c=2b(\mathrm{mod}\ n)$ where b is the integer assigned to the overcrossing and a and c are the integers assigned to the other two arcs; and

2. at least 2 distinct integers mod n are used in the diagram.

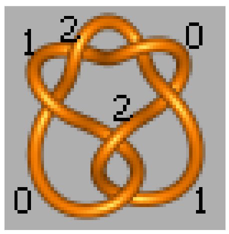


Figure 9 A Knot Coloring

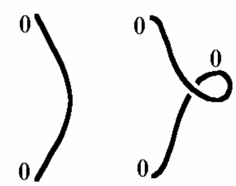
In short, a knot diagram is colorable if each crossing is either uniformly colored or has three distinct colors. Since every diagram can be uniformly colored, at least two colors must be used to show if the diagram is colorable. Figure 9 shows an appropriate coloring for a knot diagram.

Theorem 3

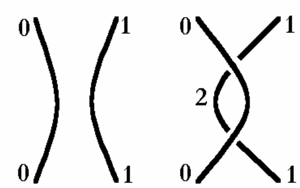
If K_1 is a knot or a link which is colorable mod n then every diagram of K_1 is colorable mod n.

Proof: The author's. In each case it must be shown that the resulting diagram is again colorable with the ends of each original arc's color unchanged. In other words, Reidemeister moves preserve colorability. Each case works from left to right, while the inverse is simply right to left.

Case 1. There is only one arrangement, which is uniformly colored, since only two arcs are made.

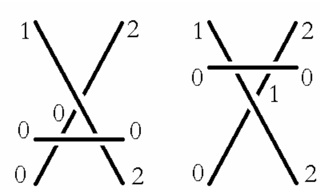


Case 2. There is only one arrangement with two (and hence three) colors. The new arc is colored with the third color.

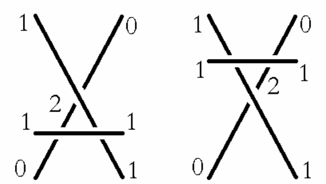


Case 3. Take a moment to note here that this Reidemeister move is equivalent for each strand. I will show, then, the various colorings, letting the inverse be apparent for each, again colorable.

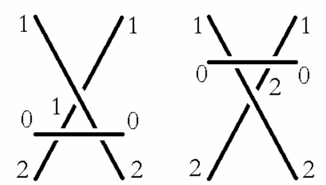
a) All that changes is the color of the short, center arc.



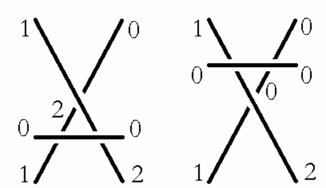
b) Here, nothing changes. Credit this to the prevailing color 1.



c) As in (a) the center arc is all that changes.



d) As in (a) the center arc is all that changes.



The reader can verify to his satisfaction whether all cases have been exhausted. Hence Reidemeister moves preserve colorability. ❑

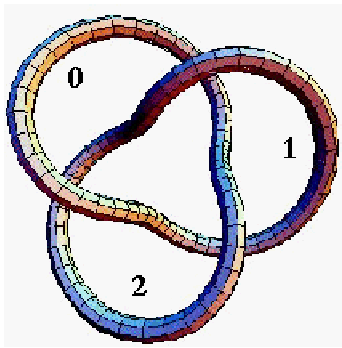
Corollary 1

From Theorem 1 and Theorem 2 it follows that if K_1 is a knot or a link which is colorable mod n and K_2 is equivalent to K_1, then K_2 is colorable mod n. ❍

Corollary 2

There exists a knot which is not equivalent to the unknot. ❍

Proof: Note that the trefoil knot is colorable mod 3 whereas the unknot is not, Figure 10. ❑



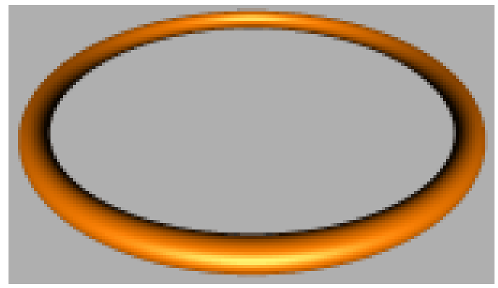


Figure 10 Two Non-Equivalent Knots

A link L is splittable if there exists a smooth (or polyhedral) 3-ball B, an ordering of the components of the link $K_1, K_2, \ldots, K_m$ and an integer $0<k<m$ such that $K_j \subset B$ for $j \leq k$ and $K_i \subset S^3 - B$ for $i > k$.

Theorem 4

If a link L is splittable then L is colorable mod 3.

Proof: Adapted from Nanyes. If L is splittable with a splitting ball B, then there exists a diagram for L in which the images of $L \cap B$ are separated from the images of $L \cap (S^3 - B)$ by a circle C. Give the components of the diagram of $L \cap B$ the monochrome coloring by assigning the integer 0 to each strand. Similarly, assign the strands of the diagram of $L \cap (S^3 - B)$ the integer 1. ❑

Chapter 3

THEOREMS RELATING TO THE BORROMEAN RINGS

Corollary 3

The Borromean Rings are not splittable. ❍

Proof: The author's. For the Borromean rings, taking advantage of the fact that $c=2b-a \pmod{n}$, a general coloring is:

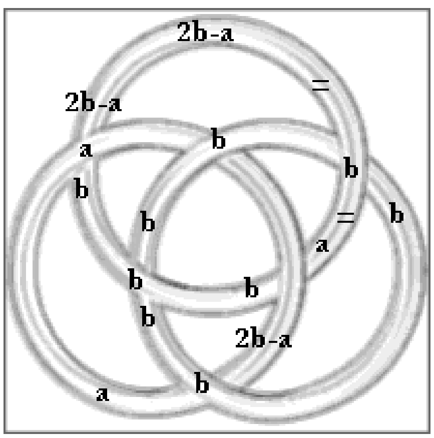


Figure 11 General Borromean Coloring

This integer labeling in Figure 11 shows that we have no choice but to set $a=b$. Thus, by the contrapositive of Theorem 4, the Borromean Rings are not splittable and thus the rings cannot be pulled apart. ❑

The arrangement of the Borromean Rings is possible with curves, which are homeomorphic to geometric circles. Here it is proved that the arrangement is impossible in $\mathfrak{R}^3$ if the circles are geometric even when the radii are arbitrary.

Theorem 5

Borromean Circles are Impossible.

Proof: Adapted from Lindström and Zetterström.

The proof is indirect. Assume the arrangement is possible using ordinary circles. Lift C_2, Figure 12, towards the spectator such that it will meet C_1 in two points in its new position ($C_{2'}$). Observe that $C_{2'}$ will not meet C_3 since we lift C_2 away from C_3.

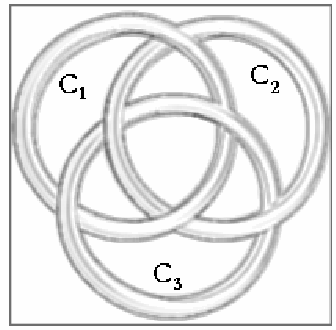


Figure 12 Three Linked Circles

There now are two cases: Either C_1 and $C_{2'}$ belong to a plane π or they do not belong to the same plane. In the first case, C_3 will be partly above and partly below plane π. If we follow circle C_3 in clockwise direction we are first above the plane, then below, then above, then below, then back again where we started above the plane. This means that the circle C_3 meets plane π at least 4 times, which is impossible for a circle not in π.

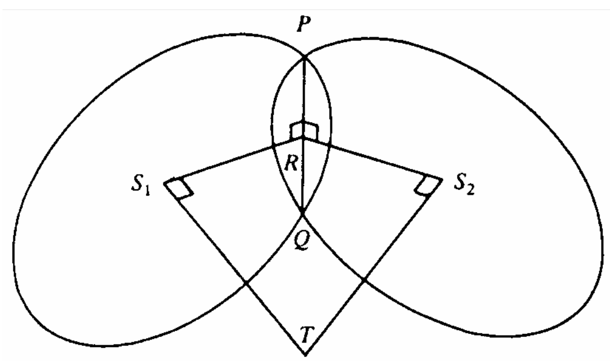


Figure 13 Two Circles on a Sphere

In the other case, when C_1 and $C_{2'}$ do not lie in the same plane, we will prove that they belong to a sphere. The circle C_3 will meet this sphere at least 4 times, again a contradiction. We will now prove that C_1 and $C_{2'}$ belong to a sphere when they are not coplanar. Let P and Q be the points in which C_1 and $C_{2'}$ meet (see Figure

13). Let R be the midpoint of the same line segment PQ. Let S_1 and S_2 be the centers of the circles C_1 and C_2, respectively. The plane π' containing R, S_1 and S_2 is orthogonal to the line PQ. Let l_i be the normal to the plane of C_i through S_i (i=1,2). The line l_i is orthogonal to the line PQ and contains the point S_i in the plane π'. It follows that l_i lies in π' (i=1,2). Since l_1 and l_2 are not parallel lines they will meet at a point T. The distance between T and any point in $C_1 \cup C_{2'}$ is a constant r. Therefore C_1 and $C_{2'}$ belong to the sphere with center T and radius r. It is not hard to see that the circle C_3 goes in and out of this sphere 4 times, which is impossible because a circle and a sphere have at most 2 points in common when the circle does not belong to the sphere. Therefore, Borromean circles are impossible! ❑

Another proof[1] of Theorem 5 goes as follows:

Three round circles in $\Re^3$ (or its conformal compactification S^3) with pairwise linking number 0 bound three mutually disjoint hemispheres in the 4-ball $H^4 \cup S^3$. Each hemisphere is the compactification of a hyperbolic plane. Consider a round 4-ball with radius $R(t)$ increasing linearly with time and with center not lying on one of the three hyperbolic planes. Let T be the last of the times when it touches one (say N) of the three planes L, M, N. Then there is a hyperplane Q tangent to the ball $B(T)$ at the point of tangency with the plane N. There is an isotopy, within the class of totally geodesic planes which are disjoint from L and M, of the plane N within the hyperplane Q that ends at a plane P' which is separated from L and M by a hyperplane disjoint from L, M and P'. Therefore a link of three round circles which have linking numbers 0 is split. (In fact by a further isotopy

[1] Thanks to Geoffrey Mess, Department of Mathematics, UCLA (geoff@math.ucla.edu) who heard this at MSRI in 1983. He thinks the argument is Freedman's. He does not think reflection groups have anything to do with it, though there may be a variant argument.

one can see that the link is the trivial link.) So the Borromean rings is not a link of round circles. ❑

C h a p t e r   4

THE BRUNNIAN EXTENSION

Can the Borromean Rings be generalized to a large number of rings? I will use β_n to represent a linkage of *n* rings which has the Borromean property; namely, that the rings are pairwise disjoint and the removal of any one ring unlinks all of the rings. Brunn has shown that one can always link *n* closed curves in such a way that by the removal of any one of these curves, the remaining *n–1* are not linked. The case which we have been working with is for *n*=3. For *n*=4, Figure 14 comes from the tract by Tietze. Each of the four curves is topologically equivalent to a circle. An equivalent diagram with metric circles is shown in Figure 15.

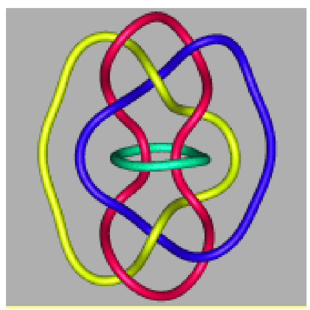


Figure 14 β_4

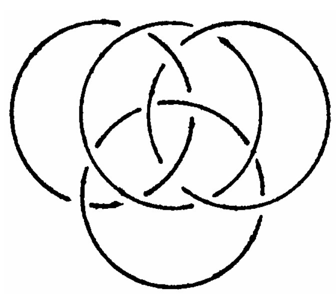


Figure 15 Metric β_4

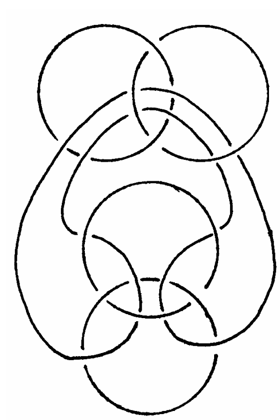


Figure 16 β_5

One solution for the problem for five circles is shown in the Figure 16. Interwoven in the four metric circles is curve *k*. It should be clear that the removal of *k* frees all the circles. Also, the removal of any circle frees *k* and the other circles.

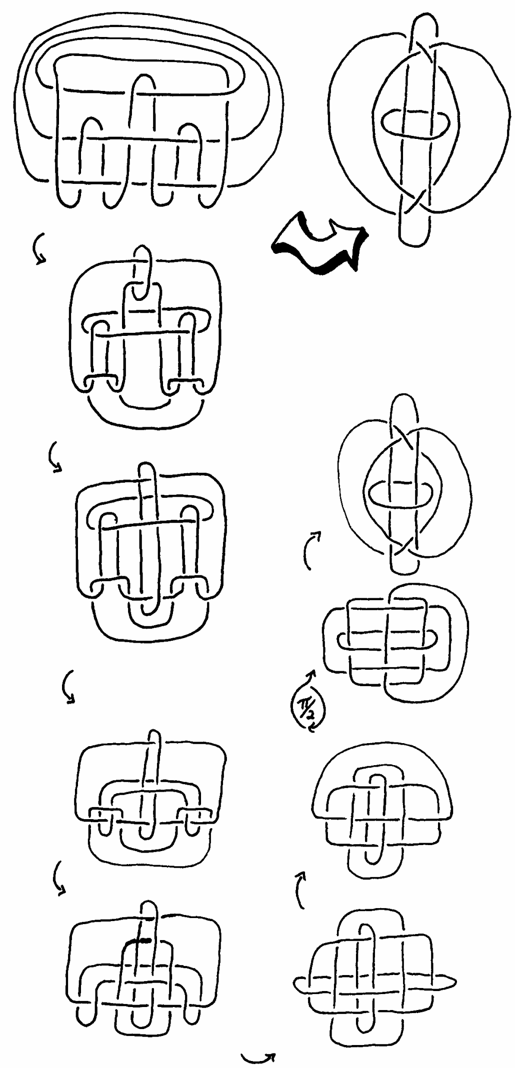


Figure 17 β_4 From General to Common Form

To establish the result for an arbitrary number of circles we will look for a general principle in the arrangement of the circles. Brunn has given such a principle. We now go over the cases for n=3,4,5 again and indicate the generalization to an arbitrary number.

Consider both Figure 14 and Figure 15. Neither the metric form nor the common form of β_4 is convenient for the sake of generalization. Consider the transformation in Figure 17. Here we have arrived at the common form of β_4 from a form which is useful for generalization.

In Figure 18 we have n–1 concentric circles and an n-th closed curve *k*, topologically equivalent to a circle, which is linked with them. It is obvious that removing curve k will free the remaining n–1 circles, since they begin unlinked. We must now show that by removing any of the concentric circles the remaining circles will then be freed; it will be sufficient to show that k is free upon removal of one circle.

Let us begin by inspecting the first diagram in Figure 18, with circles 1 and 2 and the curve k. To distinguish crossings, we will put a + for the case when k crosses a circle, and a − for the other case. Then for circle 1 we have the pattern

++−−

and for circle 2 the pattern

+−−+.

In the case for n=4 the circles 1, 2 and 3 are linked by the curve k, shown by the second diagram in Figure 18.

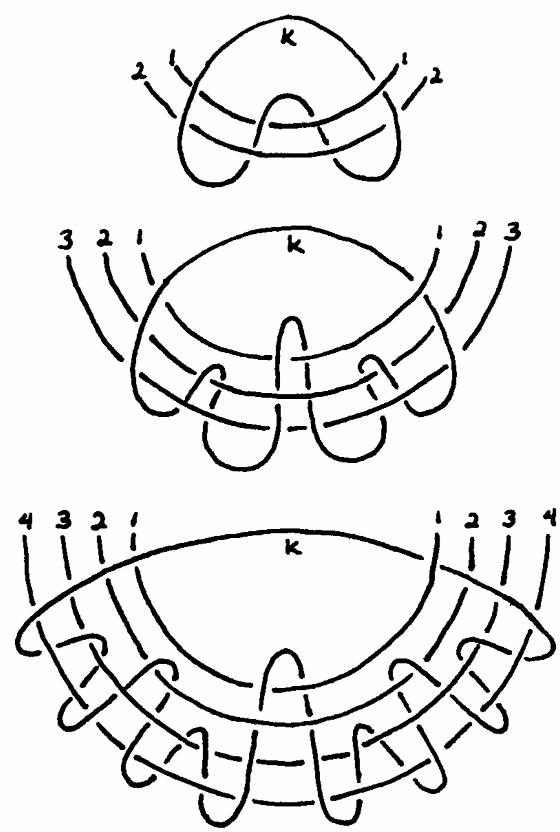


Figure 18
Brunn's
Extension

Let the first line of the crossing pattern represent circle 1, the second circle 2 and the third circle 3. The pattern now is

```
+   + −   −
++−− −−++
+−−+ +−−+.
```

Note the symmetry in the second and third rows.

The last diagram in Figure 18 is for the case n=5. The crossing pattern is now

```
+       + −       −
+  + −  − −  − +  +
++−− −−++ ++−− −−++
+−−+ +−−+ +−−+ +−−+.
```

Here the symmetric arrangement (and fractalness) is apparent. If a fifth circle is added the interlacing with circles 1 to 3 is not changed, only the 4th is altered. Where there was a + crossing of circle 4 a mesh is inserted which has ++ crossings with circle 4, and +− crossings with circle 5; Brunn says that the curve k 'bestrides' 5. Where there was a − crossing of circle 4 by k the mesh inserted there has −− crossings with that circle, and −+ crossings with 5. It is now clockwork to develop the crossing pattern for n=6,7,etc., and the reader may write these down if so inspired.

Now we must show that the removal of one of the n–1 circles frees k, and hence all the circles. We will investigate the case for n=5 (last diagram in Figure 18); it should then be clear for the general case.

If one removes circle 4 then it is immediately obvious that one can fold the meshes upwards, either at the back or the front, so that circle 3 is also freed. Once this is done, the same can be done with circle 2 and finally with circle 1.

If one removes circle 3 then all the bestriding forms +− and −+ disappear. Thereby circle 4 is freed and then, as before, circles 2 and 1.

If one removes circle 2 and thereby removes its bestriding forms, then the bestriding forms which hinder the freeing of circle 3 and 4 also fall away, and these circles become free, followed by circle 1.

If one removes circle 1 then, as with the case for circle 2, its bestriding forms which hinder the freeing of circle 2, 3 and 4 also fall away, and these circles become free.

The immediate generalization of the process to n circles is equally clear. ❑

Chapter 5

OTHER POINTS

You will have by this time asked yourself some questions. Why did he abandon his primary ideas on the subject? Why is it that knots are limited to three-space? Are there any applications for the Borromean Rings or are these rings simply another mathematical curiosity?

1. Why did I abandon my primary ideas on the subject?

To the point, the presumptions I made in my initial enthusiasm and limited experience were cheeky, and wrong. (Even Ludwig Wittgenstein discovered he was on the wrong path and did an about-face.) It was after my year of independent experimentation that I finally consulted some texts on the subject. I slowly came to realize why I was wrong in my thinking. I proceeded to tear down the architecture I had fortified and to start afresh from texts and modified ideas from the old blueprint. This thesis represents a new architecture.

2. Why is it that all these knots are limited to three-space?

Theorem 6
There are no knotted loops in four-space.

Proof: Adapted from Adams.

We are all familiar with the idea that we can consider the fourth dimension time. However, this is simply a convention for physicists since we live in a world of three spatial dimensions. If we lived in an eight-dimensional world, time would be considered the ninth dimension and hide-and-go-seek would be an entirely different game. If I want to consider an *n*-dimensional world allowing time to continue normally, I just rename time the ($n+1$) dimension, one beyond the number of spatial dimensions.

Adams was the second to describe a visualization of the fourth dimension that I was able to conquer (I found my first as I read Hawking). He describes the first three dimensions again as spatial, but lets the fourth dimension become a **color dimension**. Instead of moving along in a direction changing position, we move along by changing color. Just as we can't make a discrete jump from Jaffrey to Rindge, NH, we can't jump from yellow to blue without going through green and all the hues in between. However, unlike a color wheel, the color dimension behaves as a normal linear dimension and does not loop back on itself. Put yourself into this four-dimensional world for a moment with a pair of fashionable color goggles. With these goggles you can change where you are by adjusting the knob on the goggles to see whichever color you want. By turning the knob from blue to violet you have traveled parallel to the color axis through infinitely many distinct three-space 'slices' of this four-space. The three-dimensional analog to this is that of traveling through infinitely many planes by moving in the third direction. Nothing new here, just a level higher.

We will use this color model of four-space to show that there are no knotted loops in four-space, i.e., that every knotted loop in four-space is equivalent to the unknot. This is why knotted loops aren't studied in four-space, but knotted spheres are. Consider a knotted loop in four-space that is completely red. If we could pass the knot through itself some number of times, we could make it into the unknot.

Consider a point p along the knot where we would like the knot to pass through itself. In the short strand containing this point we will change the color to orange. We are stretching this section of the knot in the fourth color direction from red to orange. The strand changes gradually from red to orange, then gradually back to red again. After the knot passes through itself at p, the orange section goes back to red again. This is shown in Figure 19 on a black and white scale. Think of black as red and white as orange. Note that this slight change in the color of a strand of the knot is an isotopy of the knot in four-space. That is, it is an elementary deformation of the knot, with the resultant knot still equivalent to the original knot.

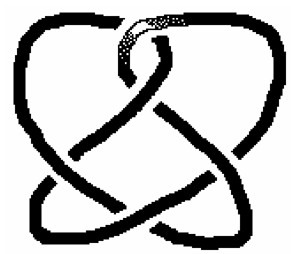


Figure 19 A Knot in Colorized Four-space

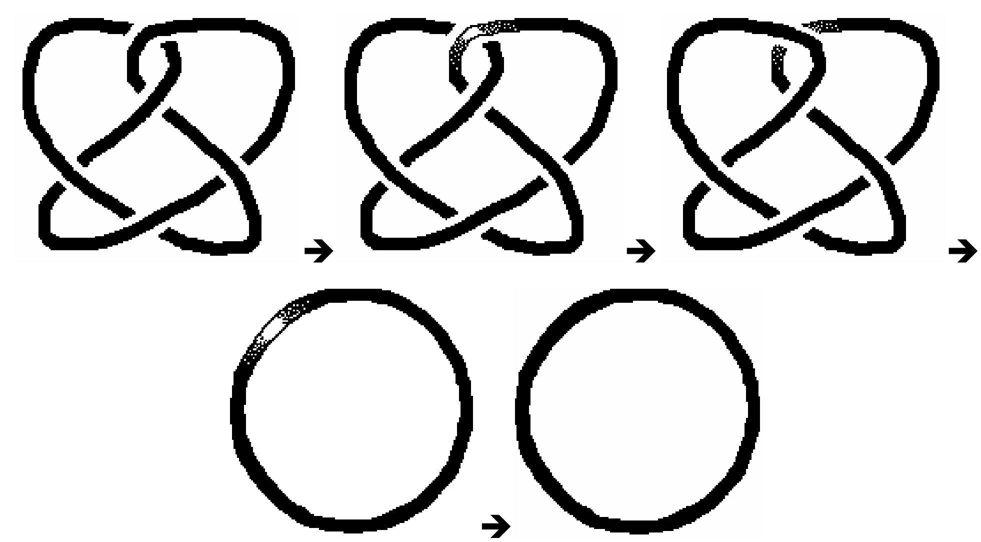


Figure 20 A Knot is Equivalent to the Unknot in Four-Space

Remember that the red and orange three-dimensional spaces are distinct. If we have our goggles adjusted to red we cannot see orange objects — they do not exist in the red three-space. Therefore we can move the red part of the knot through the orange strand. The two strands cannot see each other, they exist in different three-dimensional slices of four-space. We can repeat this operation until we have passed the appropriate strands through one another in order to unknot the knot. Then, we can push the orange strands back into the red three-dimensional world. The result is a red unknot shown in Figure 20. Also, any knotted loop in four-space is equivalent to the unknot in four-space. The study of knotted loops in four-space is completely boring, since there is only one such loop, the unknot.

3. Are there any applications or are these rings just another mathematical curiosity?

Though the Borromean rings don't appear specifically in many areas of science, knot theory is abundant. Much of the initial interest in knot theory was motivated by the possibilities of applications to chemistry. However, it wasn't until the 1980s that applications to chemistry were actually realized. Here I refer the reader to Adams since he dedicates a chapter to *Biology, Chemistry, and Physics*.

The Department of Theoretical Physics at the University of Bergen has a web site dedicated to Radioactive Nuclear Beam Theory and the implications of the Borromean Rings. The RNBT work has so far focused on few-body theory for light halo-like nuclei: Borromean systems. The 3-body quantum analog is one where the 3-body system is bound, but where none of the two-body subsystems are bound. (The following I don't grasp. I include it for those wanting to know where to look for more.)

Nature has realized Borromean systems in loosely bound halo-like nuclei such as 11Li, which are now produced as secondary beams. To understand these systems one needs to go beyond the mean field approach, to few-body theoretical procedures such as expansion on hyperspherical harmonics or the coordinate space Faddeev approach. Nuclei like 6He and 11Li can (a realistic starting point!) be modeled as a core surrounded by two loosely bound valence neutrons. Not only the bound state (6He and 11Li both have only one bound state!), but also the structure of the continuum and the details of reaction mechanisms are being studied. The knowledge is important for astrophysics, and the role of halo nuclei in nucleosynthesis in supernovae is currently being investigated.

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