The Search for the Optimal Paper Helicopter

Erik Erhardt and Hantao Mai

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1 Executive Summary

Response Surface Methodology (RSM) is a powerful method to parsimoniously explore and optimize a process. In this paper we discuss the maximization of the flight time for a paper helicopter. Through the use of the statistical and mathematical techniques collected in RSM, we begin with an existing helicopter design and optimize the flight time.

To begin, we identify eight factors that potentially influence the flight time. We use an efficient screening experiment to select those vital few factors that have a significant effect on the response. The four surviving factors determine the paper weight and rotor configurations of our helicopter. This reduction in factors greatly simplifies our model.

Since the first-order model is appropriate in our initial experimental region, we use steepest ascent to move quickly to the 'near optimum' region. Curvature is detected in only a few steps, forcing us to abandon this technique in favor of a second-order model.

The analysis of this second-order model results in a stationary point which is a point of maximum response within the experimental region. We construct a confidence interval for the mean response predicted by this stationary point as well as a confidence region for the stationary point. By conducting the confirmatory experiment, we find that the observed mean of the responses at the stationary point is well within the 95% confidence interval of the mean response. This confirms the validity of our prediction.

Finally, we provide the pattern for the optimized helicopter in the Appendix.

2 Problem Description

The goal of this project is to use response surface methodology (RSM) to maximize the flight time of a paper helicopter. We employ experimental strategies for exploring the interest region in the independent variables, statistical modeling to develop appropriate relationships between the response and the variables, and optimization methods to find the levels in the variables which maximize our response (in our case, the flight time).

We begin our work by choosing a pattern for our helicopter. We decide to use the pattern provided by Prof. Petruccelli, given in Appendix A.7. This is an easy pattern to modify and replicate.

We then identify the many factors which may contribute to the flight time of the helicopter. These factors are included in the *Helicopter Factors Table* on page 5.

Table 1: Helicopter Factors

A=Rotor Length
B=Rotor Width
C=Body Length
D=Foot Length
E=Fold Length
F=Fold Width
G=Paper weight
H=Direction of fold

We decided to retain all of these factors for experimentation. At the outset, it is better to have too many rather than too few factors.

In order to determine the vital few factors that contribute to flight time we perform a screening experiment using a 2^k fractional factorial design with the initial helicopter pattern as our center point.

It is likely that our initial helicopter pattern begins our experimentation far from the region of optimum conditions. We expect at this stage that first-order approximation might be quite reasonable. We will test for curvature, but expect it will not be significant. We will perform the steepest ascent procedure to move quickly to the 'near optimum' region. As we approach the region of optimum conditions, we expect that curvature will be more prevalent and that the interactions among the factors will cease to be negligible. At that point we will test for curvature. If curvature is found, we will fit a complete second-order model to locate the stationary point (we expect a maximum) and conduct confirmatory experiments. If we do not find a maximum, ridge analysis will be useful to find a maximum within the experimental region.

3 Screening Experiment

3.1 Data Generation

In order to identify those factors important in keeping the helicopter aloft, we will construct an efficient screening design. Our pattern has 8 factors. We desire to use a fractional factorial design that has at least resolution IV.

A resolution IV design has the advantage that no main effect is aliased with any other main effect or two-factor interaction. Even though two-factor interactions are aliased with each other, we expect this design will give us sufficient information on the significant main effects.

We use a 2_{IV}^{8-4} design. This reduces significantly the number of runs while retaining resolution IV. The Confounding Rules [1] for this design are given in *Factor Confounding Rules Table* on page 6 with the complete aliasing structure in Appendix A.1.

Table 2: Factor Confounding Rules

E=BCD
F=ACD
G=ABD
H=ABC

These confounding rules define the experimental design points listed in the *Design Points for* 2_{IV}^{8-4} *Design Screening Experiment Table* on page 7.

3.2 Experimental Protocol

3.2.1 Blocking Considerations

Before creating and dropping our helicopters we define conditions under which blocking is a necessary consideration. For helicopter creation, considerations include the paper stock, who marks the pattern on the paper, who cuts the paper and who folds the paper. For helicopter dropping, considerations include the conditions of the day (atmospheric pressure, temperature, dew point), conditions of the time (whether processes such as climate control are running, movement by people or other immediate events effecting air circulation), drop location, who drops the helicopters, dropping method and who records the time.

We wish to resolve the blocking issues as much as possible caused by potential non-homogeneous conditions. We set up our experimental protocol to eliminate as many conditional factors as possible.

For the factors we can control, we are as consistent as possible. For example, Hantao plots the helicopter patterns on the paper and Erik cuts and folds them.

[1] These confounding rules are the same as in [1] except the book switches G and H in the Confounding rules. We used SAS macro DESIGN2 to create the aliasing structure.

Table 3: Design Points for 2_{IV}^{8-4} Design Screening Experiment

Obs	BLOCK	A	B	C	D	E	F	G	H
1	1	-1	-1	-1	-1	-1	-1	-1	-1
2	1	-1	-1	-1	1	1	1	1	-1
3	1	-1	-1	1	-1	1	1	-1	1
4	1	-1	-1	1	1	-1	-1	1	1
5	1	-1	1	-1	-1	1	-1	1	1
6	1	-1	1	-1	1	-1	1	-1	1
7	1	-1	1	1	-1	-1	1	1	-1
8	1	-1	1	1	1	1	-1	-1	-1
9	1	1	-1	-1	-1	-1	1	1	1
10	1	1	-1	-1	1	1	-1	-1	1
11	1	1	-1	1	-1	1	-1	1	-1
12	1	1	-1	1	1	-1	1	-1	-1
13	1	1	1	-1	-1	1	1	-1	-1
14	1	1	1	-1	1	-1	-1	1	-1
15	1	1	1	1	-1	-1	-1	-1	1
16	1	1	1	1	1	1	1	1	1

Hantao drops the helicopters while Erik records the time. We always drop the helicopters from the second floor balcony in Fuller Laboratories at WPI. Hantao releases the helicopters while Erik times the drop with a stopwatch from the ground floor. If a helicopter experiences a disturbance during a drop, such as hitting a wall, or any other conditions which effects its performance, we redrop that helicopter. We limit ourselves to dropping only when the conditions are observed to be homogeneous, though we have no way of measuring these conditions.

By following these experimental protocols we can expect a relatively homogeneous environment while conducting our experiment. Under that situation, block is no longer a necessary consideration.

3.2.2 Tools

In the process of manufacturing our helicopters, we use only those tools which will provide the most consistent results. For layout, we use a long ruler measuring centimeters along with a liquid ink pen for non-impressioning marking. For cutting, we always use the same paper cutter.

3.2.3 Techniques

While manufacturing of the helicopters we try to limit the error in each measuring, cutting and folding to no more than 0.1 centimeters. If we produce a

helicopter not meeting these standards, we do not use it and produce a replacement.

3.2.4 Factor Levels

Considerable attention is placed on the levels of each factor with variable screening and process optimization as our goals. Choosing these levels on each factor is a vital decision and something we do not take lightly. We understand that sloppy decisions in these early stages may lead to the inability to identify significant factors and, at later stages, to inefficient process optimization. However, we also realize the difficulty in choosing ranges should not prohibit the use of a screening design and region seeking methods.

In considering our design, there are some inherent constraints between factors. One relationship is between fold width and rotor width. When the rotor width increases, the fold width does not increase proportionately with the total width. So levels are chosen carefully in these two factors so that if they are indeed important, they are not judged to be insignificant in the analysis due to poorly chosen levels.

Another factor constraint is that the largest foot length does not exceed $\frac{1}{2}$ the smallest fold length. This provides the maximum availability for coded extremes in these two factors.

Also, two of our factors, G[2] and H[3] , are not continuous in their levels.

In all other factors we attempt to provide an appropriate range with hope to find the significant factors in the screening experiment.

We start with a model approximately the size of the pattern given in Appendix A.7. To this model we assigned the natural units to coded values of 0. Then, by the above considerations we decide to choose the range which seems appropriate to this helicopter pattern. Then we assign the values of the natural units at the high and low levels of the range to the coded values of -1 and +1. This can be seen in the *Factor Level in Coded and Natural Units Table* on page 9.

3.3 Collection of Data

Based on the experimental protocol above, we cut and fold 16 helicopters in random order at the design points listed in the *Design Points for* 2_{IV}^{8-4} *Design Screening Experiment Table* on page 7. We carefully transport them to Fuller Laboratories to conduct our experiment. We drop the helicopters in random order recording the drop time of each. Our results for the Screening drop are given in the *Screening Drop Results Table* on page 9. Accompanying SAS Code for the Screening Drop is included in Appendix A.2

[2]Since Paper weight is not a continuous factor, we confine ourselves to two levels. Light is from phone book whitepages and heavy is standard copy paper.

[3]Since Direction of fold is not a continuous factor, we confine ourselves to two levels. Against indicates the fold direction is opposite the direction of rotation. For example, if the rotor on right side of the helicopter folds away from you, the fold on the right side will fold toward you.

Table 4: Factor levels in Coded and Natural Units

	Coded Values			
Factors	-1	0	1	Range
A	5.5	8.5	11.5	6
B	3	4	5	2
C	1.5	3.5	5.5	4
D	0	1.25	2.5	2.5
E	5	8	11	6
F	1.5	2	2.5	1
G	light	(none)	heavy	(none)
H	against	(none)	with	(none)

Table 5: Screening Drop Results

Obs	Ord	A	B	C	D	E	F	G	H	Time
1	12	-1	-1	-1	-1	-1	-1	-1	-1	11.80
2	7	-1	-1	-1	1	1	1	1	-1	8.29
3	11	-1	-1	1	-1	1	1	-1	1	9.00
4	15	-1	-1	1	1	-1	-1	1	1	7.21
5	1	-1	1	-1	-1	1	-1	1	1	6.65
6	4	-1	1	-1	1	-1	1	-1	1	10.26
7	16	-1	1	1	-1	-1	1	1	-1	7.98
8	8	-1	1	1	1	1	-1	-1	-1	8.06
9	3	1	-1	-1	-1	-1	1	1	1	9.20
10	10	1	-1	-1	1	1	-1	-1	1	19.35
11	9	1	-1	1	-1	1	-1	1	-1	12.08
12	5	1	-1	1	1	-1	1	-1	-1	20.50
13	6	1	1	-1	-1	1	1	-1	-1	13.58
14	13	1	1	-1	1	-1	-1	1	-1	7.47
15	14	1	1	1	-1	-1	-1	-1	1	9.79
16	2	1	1	1	1	1	1	1	1	9.20

3.4 Data Analysis

3.4.1 Effects Analysis

After we collect our data, we check for significance in three ways. First we look at the Normal Quantile Plot of the effects. These results are summarized in the *Normal Quantile Plot* on page 10.

Next we use SAS macro EFFECTS[4] to compare the factor effect estimates with the MOE and SMOE at the .95 confidence level. These results are sum-

[4] EFFECTS is provided in the WPI SAS macro library.

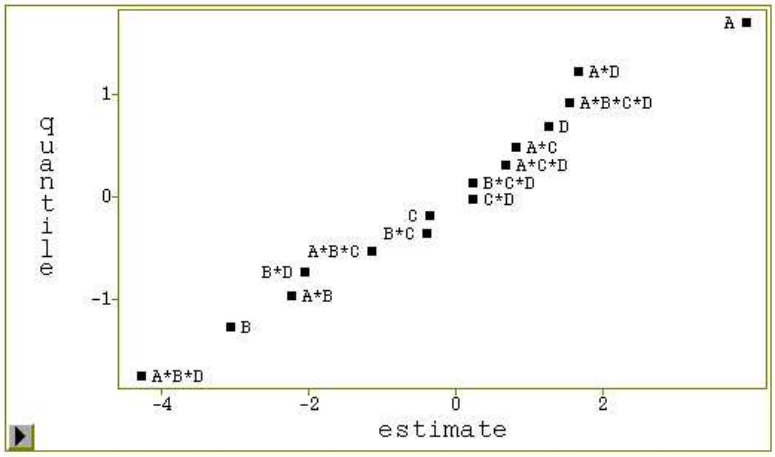

Figure 1: Normal Quantile Plot

marized in the *EFFECTS Plot* on page 11 with the accompanying *EFFECTS Test Results Table* on page 11.

Lastly, we perform the LOUGHIN permutation test, at the .90 confidence level. These results are summarized in the *LOUGHIN Permutation Test Results Table* on page 12.

From each of these methods, no effect is found to be significant. So additional center runs may be necessary.

3.4.2 Design Augmentation to Find Significant Factors

Because the above three methods did not provide significant factors, it is necessary to run some replicates at the center of the design to get an estimate of pure error. To do this we run two sets of center points, each having three runs. One set at the high and the other set at the low level of the paper weight. This will provide enough degrees of freedom to get an estimate of pure error. We expect the MOE and SMOE will decrease and we will be able to find significant factors.

Because our center points are not at the actual continuous center, it is necessary to compute the MOE and SMOE values by hand. These equations are given as (1), (2) and (3).

$$MOE = S_{PE} \times t_{c-1,(1+L)/2} \tag{1}$$

$$SMOE = S_{PE} \times t_{c-1,\delta} \tag{2}$$

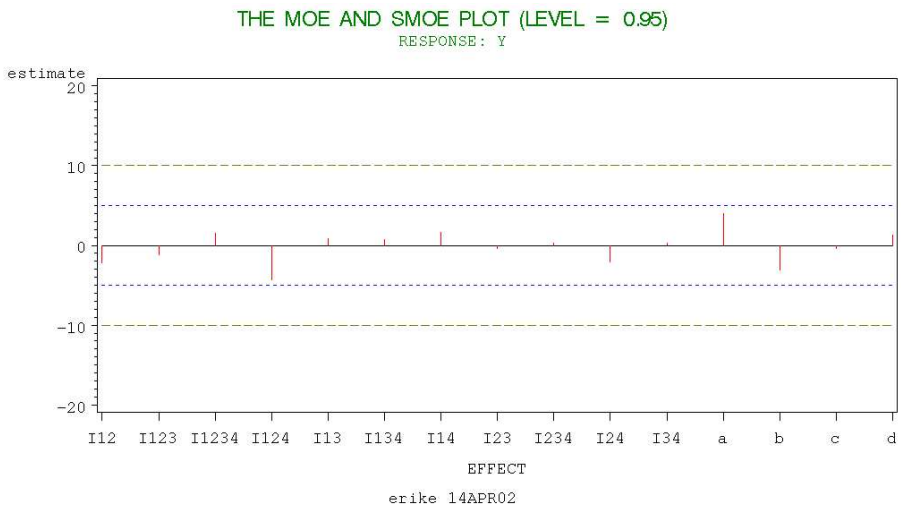

Figure 2: EFFECTS Plot

Table 6: EFFECTS Test Results

Effect	Label	Estimate	MOE	SMOE
I12	A*B	-2.2175	4.94516	10.0394
I123	A*B*C	-1.1375	4.94516	10.0394
I1234	A*B*C*D	1.5625	4.94516	10.0394
I124	A*B*D	-4.2825	4.94516	10.0394
I13	A*C	0.8400	4.94516	10.0394
I134	A*C*D	0.7000	4.94516	10.0394
I14	A*D	1.6850	4.94516	10.0394
I23	B*C	-0.3850	4.94516	10.0394
I234	B*C*D	0.2500	4.94516	10.0394
I24	B*D	-2.0350	4.94516	10.0394
I34	C*D	0.2475	4.94516	10.0394
A	A	3.9900	4.94516	10.0394
B	B	-3.0550	4.94516	10.0394
C	C	-0.3475	4.94516	10.0394
D	D	1.2825	4.94516	10.0394

$$S_{PE} = \sqrt{\frac{\sum_{i=1}^{n_1}(X_{i1} - \bar{X}_1)^2 + \sum_{i=1}^{n_2}(X_{i2} - \bar{X}_2)^2}{(n_1 - 1) + (n_2 - 1)}} \tag{3}$$

Table 7: LOUGHIN Permutation Test Results

Effect	ID	P	Decision
10.65125	0	1.0000	Reject
0.2475	12	1.0000	Reject
0.25	123	1.0000	Reject
-0.3475	2	0.8723	Reject
-0.385	23	1.0000	Reject
0.7	124	0.5433	Reject
0.84	24	0.7849	Reject
-1.1375	234	0.6220	Reject
1.2825	1	0.7417	Reject
1.5625	1234	0.6849	Reject
1.685	14	0.8017	Reject
-2.035	13	0.6985	Reject
-2.2175	34	0.8004	Reject
-3.055	3	0.4073	Reject
3.99	4	0.2216	Reject
-4.2825	134	0.2835	Reject

Below we will set each factor to coded value zero, except G and H. For reasons of parsimony, we will run H at only its low level. We expect the experiment will be sufficient to identify the significant factors. The result of the center runs are given in the *Screening Experiment Center Points Results* on page 12.

Table 8: Screening Experiment Center Points Results

Obs	Ord	A	B	C	D	E	F	G	H	Time
1	5	0	0	0	0	0	0	-1	-1	10.52
2	1	0	0	0	0	0	0	-1	-1	10.81
3	3	0	0	0	0	0	0	-1	-1	10.89
4	2	0	0	0	0	0	0	1	-1	15.91
5	4	0	0	0	0	0	0	1	-1	16.08
6	6	0	0	0	0	0	0	1	-1	13.88

By applying Equation (3) to these results, we obtain $S_{PE} = .8764$. From Equation (1), the $MOE_{.05} = 2.4332$ at the .05 significance level. By comparing the MOE to the effect estimates, we conclude that the main effects A, B and G are significant, while the others are not.

Hereafter, we will confine ourselves to the use of G at -1 (the light paper), since G has only two levels and it is clear that the response is much higher at its low level. We will also confine ourselves to the low level in H since the response

tends to have better performance at this level.

With the reduction of factors through lack of significance (C, D, E, F and H) and confinement (G and H), we have only two factors to consider (A and B). Hereafter, we can use the full 2^2 factorial replicated design to start our optimization process.

4 Optimization

4.1 Path of Steepest Ascent

4.1.1 Data Generation

Since we have obtained the vital few factors responsible for keeping the helicopter aloft, it is time to begin searching for optimum settings for those factors. By using the method of steepest ascent we can move parsimoniously to a region in which the process is improved. This method uses sequential experimentation along the path of steepest ascent so that we may expect the maximum increase in response. The path is determined by fitting a first-order model using an orthogonal design and moving in the direction of the regression coefficient estimates.

In our case we have only two factors, so it is not expensive to use a replicated design. We will use the reduction given in the previous section to define a full 2^2 replicated design by confining the values of G and H to the low levels. We use the previously folded helicopters made from light paper (G = -1), modifying any with fold direction (H) at the high level to the low level. This allows us to populate half our experiment with previously manufactured helicopters. We fold eight new helicopters to complete the 2^2 design with four replicates, and an additional three for center runs.

These two blocks of helicopters, the old ones and new ones, have different characteristics in the insignificant factors. However, the insignificant factors (C, D, E, F and H) have little effect on the response, therefore we should have comparable performance from helicopters with the same levels in the significant factors.

Our results from the 2^2 Design with four replicates for computing the path of steepest ascent are given in the 2^2 *Replicated Design Table* on page 15. Accompanying SAS Code for 2^2 Replicated Design is included in Appendix A.3.

4.1.2 Testing for Curvature

Since we include the 3 center runs in our experiment, we can check for curvature of the surface being modeled.

The idea for the test for curvature is that if the surface is not curved, then the mean response at the center point is the same as the mean of the responses at the factorial points. A measure of the mean of the responses at the factorial points is the sample mean of the responses at those points $\bar{Y}$. A measure of the mean response at the center point is the sample mean of the responses there, $\bar{Y}_0$. Thus, an estimate of the curvature is $\bar{Y} - \bar{Y}_0$. If this estimate of curvature is large relative to experimental error, there will be strong evidence that the response surface is curved.

In our case, the estimated standard error of $\bar{Y} - \bar{Y}_0$ is

$$\hat{\sigma}(\bar{Y} - \bar{Y}_0) = \sqrt{S^2_{PE}(\frac{1}{n_f} + \frac{1}{n_c})} = \sqrt{0.8464(\frac{1}{16} + \frac{1}{3})} = 0.5788 \tag{4}$$

Table 9: 2^2 Replicated Design Results

Obs	A	B	Time
1	-1	-1	10.24
2	-1	-1	9.11
3	1	-1	16.52
4	1	-1	16.99
5	-1	1	10.20
6	-1	1	9.26
7	1	1	10.02
8	1	1	9.94
9	-1	-1	11.31
10	-1	-1	10.94
11	1	-1	12.58
12	1	-1	13.86
13	-1	1	8.20
14	-1	1	9.92
15	1	1	9.95
16	1	1	9.93
17	0	0	11.67
18	0	0	10.74
19	0	0	9.83

For $L = 0.95$,

$$\hat{\sigma}(\bar{Y} - \bar{Y}_0) \times t_{c-1,(1+L)/2} = 0.5788 \times 2.92 = 1.69 > |\bar{Y} - \bar{Y}_0| = 0.4389 \tag{5}$$

so there is insufficient evidence to conclude that there is surface curvature. Therefore, we can proceed in the path of steepest ascent.

4.1.3 Direction of Steepest Ascent

The fitted regression model is given by

$$\hat{y} = 11.1163 + 1.2881a - 1.5081b \tag{6}$$

The basis for computation of points along the path of steepest ascent are chosen to be 1 cm in factor A. This corresponds to $\frac{1}{3}$ design units. As a result, the corresponding movement in design variable B is $(-1.5081/1.2881)(1/3) = -0.39$ in design units. The natural units for B is $-0.39 \times 1 = -0.39cm$.

4.1.4 Pilot Study for Choosing Insignificant Factors

Before conducting steepest ascent, we perform a pilot study in the insignificant factors in an attempt to stabilize the helicopters. We know by experience that

the factors determined insignificant do, in fact, effect the flight of the helicopter. Therefore, our desire is to find a rough optimum for these insignificant factors at the base level plus 1Δ and 2Δ in significant factors. Once these factors are set, we will continue with our steepest ascent experiment. This process is primarily done by setting factor C to 2 and varying levels of factors D, E and F in regions our experience suggests are optimal.

Steepest Ascent Preparation Experiments

Using steepest ascent design points Base+1Δ and Base+2Δ, we perform an experiment on factors D, E and F. These results are summarized in the *Steepest Ascent Preparation Experiment 1 Results Table* on page 16.

Table 10: Steepest Ascent Preparation Experiment 1 Results

Factors			Response Time	
D	E	F	Base+1Δ	Base+2Δ
0	9	1	2.98	3.30
2	9	1	3.77	3.64
4	9	1	3.35	3.95
5	9	1	3.43	3.91
4	9	2	3.82	4.07
2	9	2	3.72	3.67
0	9	2	3.36	3.64

Through Experiment 1 we find factor F (fold width) is preferable at 2cm. We perform a second experiment to gain more information about factors D and E. These results are summarized in the *Steepest Ascent Preparation Experiment 2 Results Table* on page 16.

Table 11: Steepest Ascent Preparation Experiment 2 Results

Factors			Response Time	
D	E	F	Base+1Δ	Base+2Δ
2	6	2	3.90	4.70
4	6	2	4.03	3.22
0	6	2	3.44 unst	3.89
0	6	2	3.26 unst	3.50
2	6	2	4.08	3.82

By the information obtained through these two experiments, we find a preferred level of $D = \frac{1}{4}E$. Also, factor E at 6cm gives a stable helicopter, so we will use that level.

Therefore, we determined favorable levels at C=2, D=2, E=6 and F=2, in

their natural units. As we follow the path of steepest ascent, we may find that factor F is large enough to wrap about the 'tail' of the helicopter. When this is the case, we will confine ourselves to F=$\frac{2}{3}B$, resulting in fold widths which divide the tail roughly into three equal parts.

4.1.5 Coordinates Along the Path of Steepest Ascent

We suspect five steps along the path of steepest ascent will bring us near the optimum region followed by a deterioration in response. The coordinates along the Path of Steepest Ascent with the responses at those values are summarized in the *Steepest Ascent Coordinates and Results in Natural Units Table* on page 17. Six runs were made along the path, starting at the base. Eventually, severe deterioration in response occurred due to the fact that the first order approximation is no longer valid. In this case a reduction was experienced after Base+3Δ. Our further follow-up experiment will involve an experimental design centered at Base+3Δ.

Table 12: Steepest Ascent Coordinates and Results in Natural Units

Run	A	B	(F)	Time	
Base	8.5	4.00	(2.0)	12.99	
Δ	1.0	-0.39			
Base + 1 Δ	9.5	3.61	(2.0)	15.22	
Base + 2 Δ	10.5	3.22	(2.0)	16.34	
Base + 3 Δ	11.5	2.83	(1.5)	18.78	optimum
Base + 4 Δ	12.5	2.44	(1.5)	17.39	
Base + 5 Δ	13.5	2.05	(1.2)	7.24	

4.2 Second-Order Response Surface

In our exploration of the second-order response surface we want to use a design that allows for estimation of interaction and quadratic terms. In our case, the design for this phase will be a spherical 2^2 Central Composite Design (CCD). The center runs allow for testing for curvature. If curvature is detected we will abandon steepest ascent. The CCD is given in the *CCD Levels in Coded and Natural Units Table* on page 18. The results are summarized in the *CCD Experiment Results Table* on page 18. Accompanying SAS Code for the CCD analysis is included in Appendix A.4.

Here, we assume the risk of performing unnecessary runs at the axial points in the case where curvature is not detected. However, in consideration of blocking and estimations of quadratic terms, the axial points will be necessary if curvature is found.

Table 13: CCD Levels in Coded and Natural Units

Factor	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$
A	10.08	10.50	11.50	12.50	12.91
B	2.28	2.44	2.83	3.22	3.38
C	2.0	2.0	2.0	2.0	2.0
D	1.5	1.5	1.5	1.5	1.5
E	6.0	6.0	6.0	6.0	6.0
F	1.5	1.5	1.5	2.0	2.0
G	light	light	light	light	light
H	against	against	against	against	against

Table 14: CCD Experiment Results

Obs	Ord	A	B	Time
1	7	-1	-1	13.65
2	3	1	-1	13.74
3	11	-1	1	15.48
4	5	1	1	13.53
5	9	0	0	17.38
6	2	0	0	16.35
7	1	0	0	16.41
8	10	1.414	0	12.51
9	4	-1.414	0	15.17
10	6	0	1.414	14.86
11	8	0	-1.414	11.85

4.2.1 Testing for Curvature

Using Equations (4) and (5) we test for response curvature 'near' the optimal region.

In our case, the estimated standard error of $\bar{Y} - \bar{Y}_0$ is

$$\hat{\sigma}(\bar{Y} - \bar{Y}_0) = \sqrt{0.3342(\frac{1}{4} + \frac{1}{3})} = 0.4415 \tag{7}$$

For $L = 0.95$,

$$\hat{\sigma}(\bar{Y} - \bar{Y}_0) \times t_{c-1,(1+L)/2} = 0.4415 \times 2.92 = 1.289 < |\bar{Y} - \bar{Y}_0| = 2.6133 \tag{8}$$

so there is sufficient evidence to conclude that there is surface curvature. This completes our path of steepest ascent and we may proceed with second-order modeling.

4.2.2 Fitting a Second-Order Response Surface Model

Using the data given in the *CCD Experiment Results Table* on page 18, the corresponding fitted second-order model is given by

$$\hat{y} = 16.713 - 0.702\,a + 0.735\,b - 1.311\,a^2 - 0.510\,ab - 1.554\,b^2 \tag{9}$$

In order to thoroughly check our model, the following three tests are performed. First, we will look at the significance in each of the model terms. Second, we perform a lack-of-fit test. Third, we verify our statistical assumptions. The complete output from the SAS RSREG Procedure is given in Appendix A.5.

T-test for Regression Coefficients From the *RSREG Parameter Estimates Table* on page 19, we find all the factors except the interaction term to be significant at the .05 level. This test gives us a very satisfactory result.

Table 15: RSREG Parameter Estimates

Parameter	DF	Estimate	Standard Error	t Value	$Pr > \|t\|$
Intercept	1	16.713333	0.407813	40.98	0.0001
a	1	-0.702726	0.249733	-2.81	0.0374
b	1	0.734598	0.249733	2.94	0.0322
a*a	1	-1.311042	0.297242	-4.41	0.0070
b*a	1	-0.510000	0.353176	-1.44	0.2083
b*b	1	-1.553542	0.297242	-5.23	0.0034

Lack-of-Fit Test The result from the lack-of-fit test is given in the *Lack-of-Fit Test Table* on page 19. Because the P-value for this test statistic is very large, we accept the hypothesis that the model adequately describes the data.

Table 16: Lack-of-Fit Test

Residual	DF	Sum of Squares	Mean Square	F Value	$Pr > F$
Lack of Fit	3	1.826200	0.608733	1.82	0.3737
Pure Error	2	0.668467	0.334233		
Total Error	5	2.494666	0.498933		

Residual Test Visual inspection of the *Normal Quantile Plot for Residuals Plot (right)* on page 20 indicates that the Normal assumption for the error is basically satisfied. By looking at the accompanying *Residuals vs.*

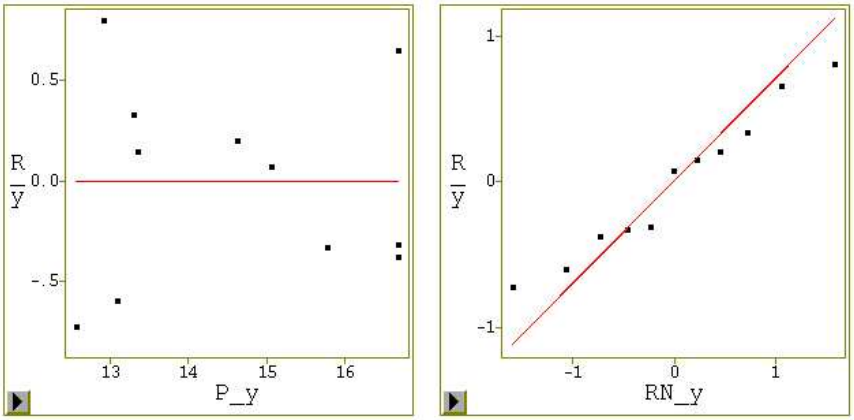

Figure 3: Residuals Plots

Predicted Value Plot (left), the constant variance is also basically satisfied. Further evidence is given by the Shapiro-Wilk test. The P-value for this test is so large that we accept the Normal assumption.

Test Statistic	Value	p-value
Shapiro-Wilk	0.959172	0.7613

Prediction of the Stationary Point

Upon finishing the model checking, we now try to use our model to predict the stationary point. The results from RSREG is given in the *RSREG Stationary Point Table* on page 21. We are pleased the stationary point is a maximum in the experimental region.

By using the tools referred by [3], we construct and present to you with great pleasure our *Response Surface* on page 22 and our *90% Confidence Region for the Stationary Point* on page 23.

The region shown in the *90% Confidence Region for the Stationary Point Figure* on page 23 depicts the 90% confidence region for the stationary point. A true response surface maximum at any location inside the 90% confidence region could readily have produced the data that was observed. The quality of the confidence region depends a great deal on the nature of the design and the fit of the model to the data. Recall from Appendix A.5 that R^2 is approximately 92%, indicating a good fit.

It should be emphasized that a confidence region that is moderate in area is not necessarily bad news. A model that fits well yet generates a confidence region such as ours on the stationary point may imply flexibility in choosing the optimum. We realize that in many RSM situations, estimation of optimum

conditions is very difficult. However, we should not get the feeling that RSM will not produce important and interesting results about the process.

Table 17: RSREG Stationary Point

The RSREG Procedure
Canonical Analysis of Response Surface

Factor	Critical Value
a	-0.324343
b	0.289665

Predicted value at stationary point: 16.933689

Eigenvalues	Eigenvectors a	Eigenvectors b
-1.149933	0.845405	-0.534126
-1.714651	0.534126	0.845405

Stationary point is a maximum.

The coded factors for the stationary point given in the *RSREG Stationary Point Table* above for factors A and B convert to the natural values given in the *Optimum Helicopter Factor Levels Table* on page 21. The pattern for our optimal helicopter is provided for your enjoyment in Appendix A.8.

Table 18: Optimum Helicopter Factor Levels

Factor	Coded	Natural
A	-0.32	11.18
B	0.29	2.94
C	x	2.0
D	x	2.0
E	x	6.0
F	x	1.5
G	light	light
H	against	against

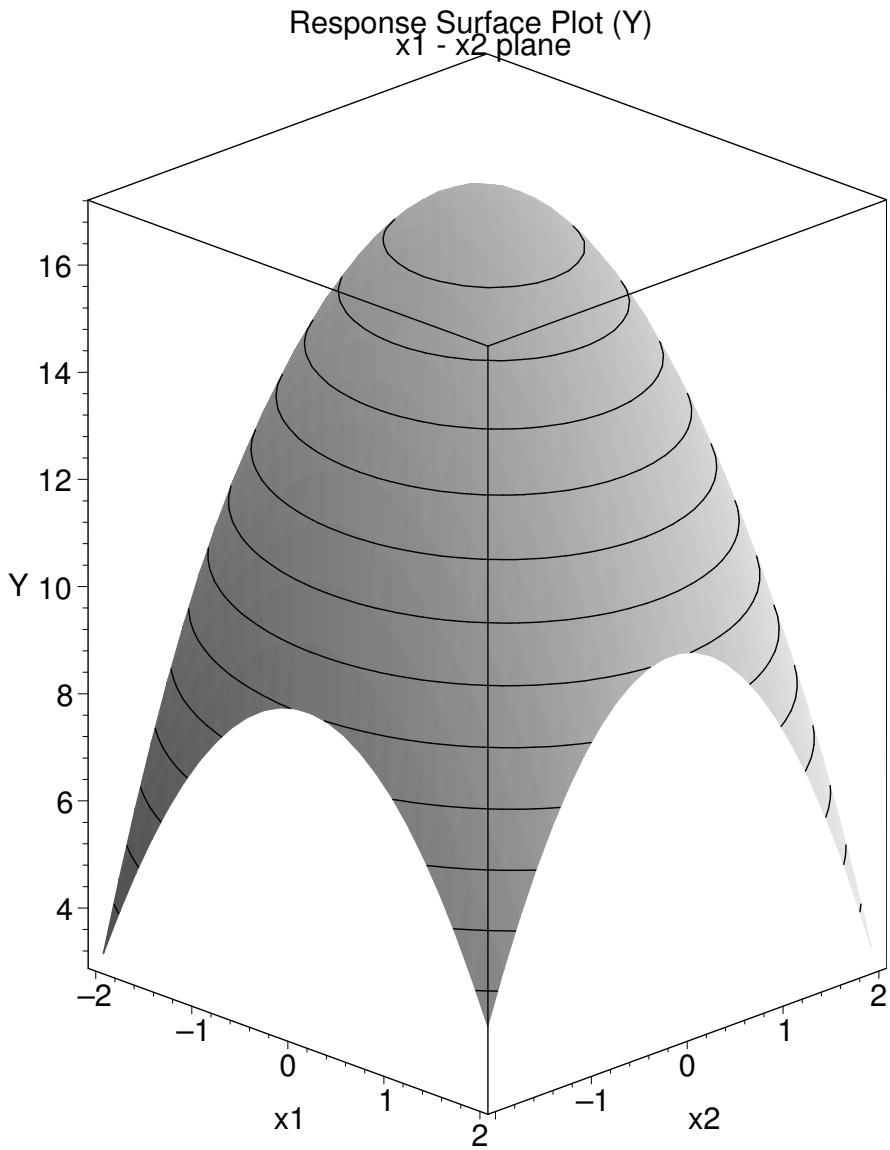

Figure 4: Response Surface

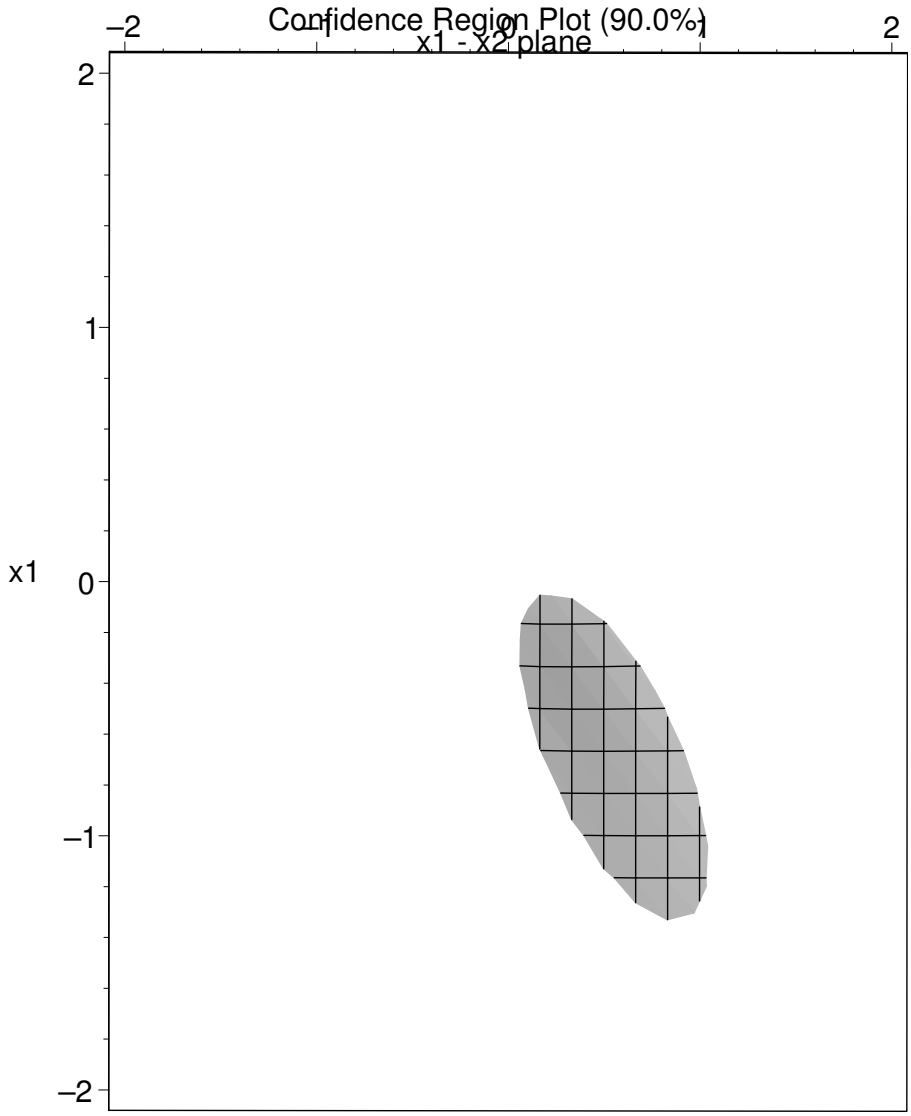

Figure 5: 90% Confidence Region for the Stationary Point

4.2.3 Confirmatory Experiment

Now that we have the setting for the optimal helicopter, we want to do a confirmatory experiment to test the validity of this result. We manufacture six helicopters according to our optimal setting and record their dropping times. These results appear in the *Confirmatory Experiment Results Table* on page 24.

Table 19: Confirmatory Experiment Results

Obs	Time
1	15.54
2	16.40
3	19.67
4	19.41
5	18.55
6	17.29

This confirmatory experiment gives a mean response of 17.8100 with standard deviation 1.6720. The 95% confidence interval on the mean response is given by [15.8314, 18.0360]. Our response falls within this interval, thus confirming the validity of our response model. Accompanying Matlab commands for *Computing Confidence Interval on the Mean Response* are included in Appendix 32.

5 Conclusions

There are several important items that have become apparent from this experience with response surface methodology. The choice of experimental region needs to be made very carefully. Choice of range in the natural levels cannot be taken lightly. This is particularly crucial in the phase of the analysis when variable screening is being done. The natural sequential nature of RSM allows us to make intelligent choices of variable ranges after preliminary phases of the study have been analyzed. In our example, following variable screening, we were careful to make good choices in the insignificant factors and well reasoned choices of levels in the significant factors for the steepest ascent procedure. After steepest ascent was accomplished, our experience gave rise to easy choices of variable levels to be used for a second-order analysis.

The goal of our experiment is to find optimum conditions in the helicopter factors to maximize flight time. However, we are careful throughout the course of our experimentation to keep in mind that estimated optimum conditions from one experiment is only an estimate. The next experiment may well find the estimated optimum conditions to be at a different set of coordinates. An example of this occurred in the screening experiment when a time of over 20 seconds was observed. This response was the greatest achieved, though the factor levels for this helicopter are far from the optimal. A comparable response was not seen again, neither by redropping the same helicopter, nor by the optimal helicopter achieved through RSM. Furthermore, as uncontrollable noise factors change from time to time, a computed set of optimum conditions may be a fleeting concept. What is more important is for the analysis to reveal important information about the process and information about the roles of the variables. The computation of a stationary point, a canonical analysis, or a ridge analysis may lead to important information about the process, and this in the long run will often be more valuable than a single set of coordinates representing an estimate of optimum conditions.

A Appendix

A.1 Aliasing Structure

```
              Factor Confounding Rules

                 E = BCD
                 F = ACD
                 G = ABD
                 H = ABC

Aliasing Structure

 I = ABCH = ABDG = ABEF = ACDF = ACEG = ADEH = AFGH = BCDE
   = BCFG = BDFH = BEGH = CDGH = CEFH = DEFG = ABCDEFGH
 A = BCH = BDG = BEF = CDF = CEG = DEH = FGH = ABCDE = ABCFG
   = ABDFH = ABEGH = ACDGH = ACEFH = ADEFG = BCDEFGH
 B = ACH = ADG = AEF = CDE = CFG = DFH = EGH = ABCDF = ABCEG
   = ABDEH = ABFGH = BCDGH = BCEFH = BDEFG = ACDEFGH
 C = ABH = ADF = AEG = BDE = BFG = DGH = EFH = ABCDG = ABCEF
   = ACDEH = ACFGH = BCDFH = BCEGH = CDEFG = ABDEFGH
 D = ABG = ACF = AEH = BCE = BFH = CGH = EFG = ABCDH = ABDEF
   = ACDEG = ADFGH = BCDFG = BDEGH = CDEFH = ABCEFGH
 E = ABF = ACG = ADH = BCD = BGH = CFH = DFG = ABCEH = ABDEG
   = ACDEF = AEFGH = BCEFG = BDEFH = CDEGH = ABCDFGH
 F = ABE = ACD = AGH = BCG = BDH = CEH = DEG = ABCFH = ABDFG
   = ACEFG = ADEFH = BCDEF = BEFGH = CDFGH = ABCDEGH
 G = ABD = ACE = AFH = BCF = BEH = CDH = DEF = ABCGH = ABEFG
   = ACDFG = ADEGH = BCDEG = BDFGH = CEFGH = ABCDEFH
 H = ABC = ADE = AFG = BDF = BEG = CDG = CEF = ABDGH = ABEFH
   = ACDFH = ACEGH = BCDEH = BCFGH = DEFGH = ABCDEFG
 AB = CH = DG = EF = ACDE = ACFG = ADFH = AEGH = BCDF = BCEG
     = BDEH = BFGH = ABCDGH = ABCEFH = ABDEFG = CDEFGH
 AC = BH = DF = EG = ABDE = ABFG = ADGH = AEFH = BCDG = BCEF
     = CDEH = CFGH = ABCDFH = ABCEGH = ACDEFG = BDEFGH
 AD = BG = CF = EH = ABCE = ABFH = ACGH = AEFG = BCDH = BDEF
     = CDEG = DFGH = ABCDFG = ABDEGH = ACDEFH = BCEFGH
 AE = BF = CG = DH = ABCD = ABGH = ACFH = ADFG = BCEH = BDEG
     = CDEF = EFGH = ABCEFG = ABDEFH = ACDEGH = BCDFGH
 AF = BE = CD = GH = ABCG = ABDH = ACEH = ADEG = BCFH = BDFG
     = CEFG = DEFH = ABCDEF = ABEFGH = ACDFGH = BCDEGH
 AG = BD = CE = FH = ABCF = ABEH = ACDH = ADEF = BCGH = BEFG
     = CDFG = DEGH = ABCDEG = ABDFGH = ACEFGH = BCDEFH
 AH = BC = DE = FG = ABDF = ABEG = ACDG = ACEF = BDGH = BEFH
     = CDFH = CEGH = ABCDEH = ABCFGH = ADEFGH = BCDEFG
```

A.2 SAS Code for Screening Drop

```
data drop1;
 input obs order a b c d e f g h y;
ab=a*b;
ac=a*c;
ad=a*d;
bc=b*c;
bd=b*d;
cd=c*d;
bcd=b*c*d;
acd=a*c*d;
abd=a*b*d;
abc=a*b*c;
abcd=a*b*c*d;
cards;
   1   12  -1  -1  -1  -1  -1  -1  -1  -1  11.80
   2    7  -1  -1  -1   1   1   1   1  -1   8.29
   3   11  -1  -1   1  -1   1   1  -1   1   9.00
   4   15  -1  -1   1   1  -1  -1   1   1   7.21
   5    1  -1   1  -1  -1   1  -1   1   1   6.65
   6    4  -1   1  -1   1  -1   1  -1   1  10.26
   7   16  -1   1   1  -1  -1   1   1  -1   7.98
   8    8  -1   1   1   1   1  -1  -1  -1   8.06
   9    3   1  -1  -1  -1  -1   1   1   1   9.20
  10   10   1  -1  -1   1   1  -1  -1   1  19.35
  11    9   1  -1   1  -1   1  -1   1  -1  12.08
  12    5   1  -1   1   1  -1   1  -1  -1  20.50
  13    6   1   1  -1  -1   1   1  -1  -1  13.58
  14   13   1   1  -1   1  -1  -1   1  -1   7.47
  15   14   1   1   1  -1  -1  -1  -1   1   9.79
  16    2   1   1   1   1   1   1   1   1   9.20
;
run;
```

A.3 SAS Code for 2^2 Replicated Design

```
data drop2;
 input obs a b y;
cards;
   1  -1  -1  10.24
   2  -1  -1   9.11
   3   1  -1  16.52
   4   1  -1  16.99
   5  -1   1  10.20
   6  -1   1   9.26
   7   1   1  10.02
   8   1   1   9.94
   9  -1  -1  11.31
  10  -1  -1  10.94
  11   1  -1  12.58
  12   1  -1  13.86
  13  -1   1   8.20
  14  -1   1   9.92
  15   1   1   9.95
  16   1   1   9.93
  17   0   0  11.67
  18   0   0  10.74
  19   0   0   9.83
;
run;
```

A.4 SAS Code for CCD Analysis

```
data drop4;
 input obs ord a b y;
 if a=9 then a=sqrt(2);
 if a=-9 then a=-sqrt(2);
 if b=9 then b=sqrt(2);
 if b=-9 then b=-sqrt(2);
cards;
   1  7  -1  -1  13.65
   2  3   1  -1  13.74
   3 11  -1   1  15.48
   4  5   1   1  13.53
   5  9   0   0  17.38
   6  2   0   0  16.35
   7  1   0   0  16.41
   8 10   9   0  12.51
   9  4  -9   0  15.17
  10  6   0   9  14.86
  11  8   0  -9  11.85
;
run;

proc rsreg data=drop4 out=drop4_out;
 model y=a b/nocode actual residual predict;
run;
```

A.5 SAS Code for RSREG Procedure

```
                        The RSREG Procedure

                   Response Surface for Variable y

               Response Mean                  14.630000
               Root MSE                        0.706352
               R-Square                          0.9167
               Coefficient of Variation          4.8281


                   Type I Sum
Regression    DF   of Squares    R-Square    F Value    Pr > F

Linear         2     8.267663      0.2761       8.29    0.0259
Quadratic      2    18.138871      0.6058      18.18    0.0051
Crossproduct   1     1.040400      0.0347       2.09    0.2083
Total Model    5    27.446934      0.9167      11.00    0.0099

                     Sum of
Residual      DF    Squares   Mean Square F Value  Pr > F

Lack of Fit    3   1.826200      0.608733    1.82  0.3737
Pure Error     2   0.668467      0.334233
Total Error    5   2.494666      0.498933

                          Standard
Parameter DF   Estimate      Error    t Value    Pr > |t|

Intercept  1  16.713333   0.407813      40.98      <.0001
a          1  -0.702726   0.249733      -2.81      0.0374
b          1   0.734598   0.249733       2.94      0.0322
a*a        1  -1.311042   0.297242      -4.41      0.0070
b*a        1  -0.510000   0.353176      -1.44      0.2083
b*b        1  -1.553542   0.297242      -5.23      0.0034


                   Sum of
  Factor DF       Squares   Mean Square    F Value    Pr > F

  a       3     14.697326      4.899109       9.82    0.0155
  b       3     18.986602      6.328867      12.68    0.0090


                        The RSREG Procedure
```

Canonical Analysis of Response Surface

Factor	Critical Value
a	-0.324343
b	0.289665

Predicted value at stationary point: 16.933689

Eigenvalues	Eigenvectors a	b
-1.149933	0.845405	-0.534126
-1.714651	0.534126	0.845405

Stationary point is a maximum.

Studentized residual tests below show not enough evidence to reject Normality assumption.

RT_y_1

N	11.0000	Sum Wgts	11.0000
Mean	0.0552	Sum	0.6071
Std Dev	1.4400	Variance	2.0735
Skewness	0.6529	Kurtosis	1.3921
USS	20.7689	CSS	20.7354
CV	2609.1854	Std Mean	0.4342

100% Max	3.1777	99.0%	3.1777
75% Q3	0.7589	97.5%	3.1777
50% Med	0.1763	95.0%	3.1777
25% Q1	-0.6915	90.0%	1.2078
0% Min	-2.2115	10.0%	-1.5290
Range	5.3892	5.0%	-2.2115
Q3-Q1	1.4504	2.5%	-2.2115
Mode	.	1.0%	-2.2115

Test Statistic	Value	P-value
Shapiro-Wilk	0.959172	0.7613
Kolmogorov-Smirnov	0.130719	>.1500
Cramer-von Mises	0.036858	>.2500
Anderson-Darling	0.257465	>.2500

A.6 Matlab Commands for Computing Confidence Interval on the Mean Response

```
x1=[-1 1 -1 1 0 0 0 -sqrt(2) sqrt(2) 0 0]'
x2=[-1 -1 1 1 0 0 0 0 0 -sqrt(2) sqrt(2)]'
x=[ones(11,1) x1 x2 x1.*x1 x2.*x2 x1.*x2]
mse=.4989
x0=[1 -0.324343 0.289665 -0.324343^2 0.289665^2 -0.324343*0.289665]'
y0= 16.933689
t=2.5706
lb=y0-t*sqrt(mse*(x0'*inv(x'*x)*x0))
ub=y0+t*sqrt(mse*(x0'*inv(x'*x)*x0))
```

A.7 Initial Helicopter Pattern

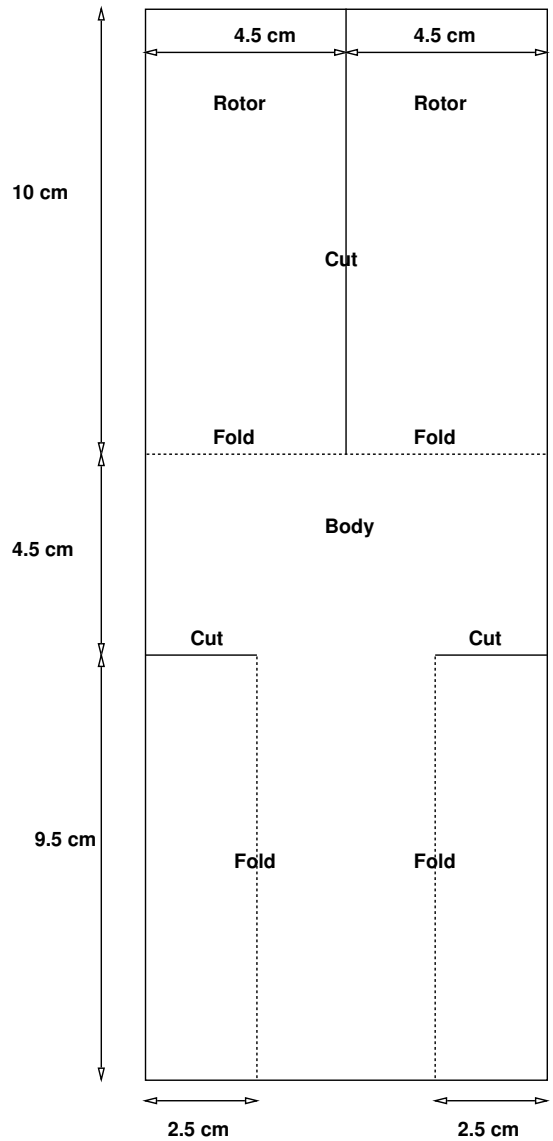

Figure 6: Initial Helicopter Pattern

A.8 Optimal Helicopter Pattern

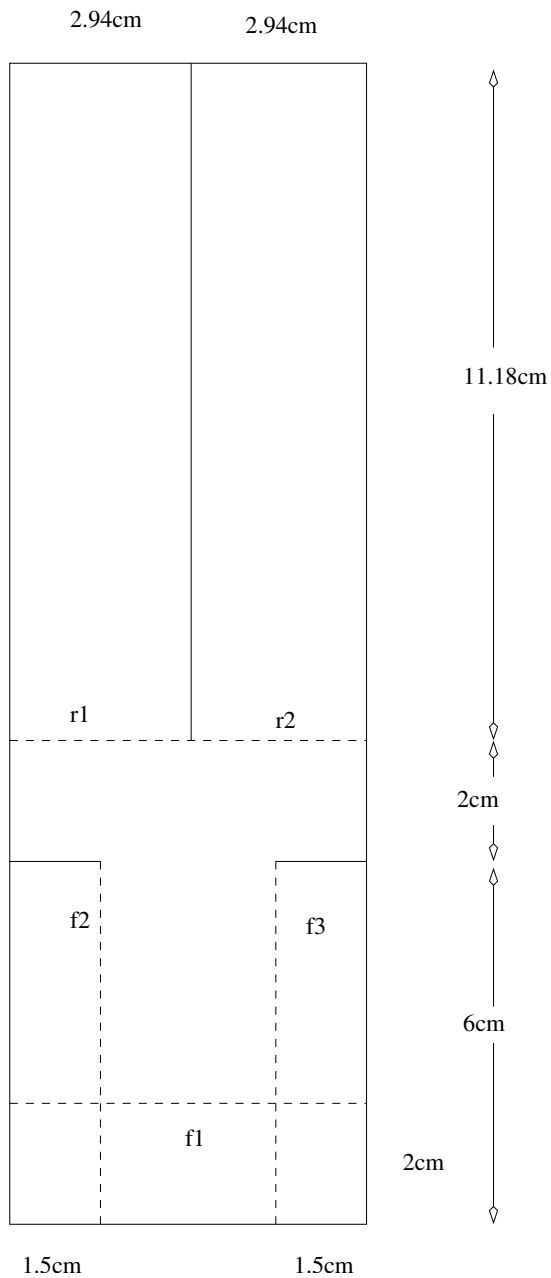

Figure 7: Optimal Helicopter Pattern

B Bibliography

References

[1] Myers, Raymond H. and Montgomery, Douglas C., *Response Surface Methodology, Process and Product Optimization Using Designed Experiments*, John Wiley & Sons, Inc., 1995.

[2] Petruccelli, J. D., Nandram, B. and Chen, M., *Applied Statistics for Engineers and Scientists*, Simon & Schuster, 1999.

[3] Del Castillo, Enrique and Cahya, Suntara. *A Tool for Computing Confidence Regions on the Stationary Point of a Response Surface*, The American Statistician, November 2001, Vol. 55, No. 4.