

Scaling Homework

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Units in the dimensional analysis exercises (1, 2, and 5) are $L =$ length, $M =$ mass, and $T =$ time.

Exercise 1 *Pendulum.*

$$\begin{aligned}t &\propto l^\alpha m^\beta g^\gamma && \text{in variables} \\[T] &= [L]^\alpha [M]^\beta [LT^{-2}]^\gamma && \text{in units}\end{aligned}$$

$\beta = 0$ to ignore mass M , $\alpha = -\gamma$ to cancel length L , $\gamma = -\frac{1}{2}$ to balance time T on both sides.

$$\begin{aligned}[T] &= [L]^{\frac{1}{2}} [M]^0 [LT^{-2}]^{-\frac{1}{2}} && \text{in units} \\t &\propto l^{\frac{1}{2}} m^0 g^{-\frac{1}{2}} = \sqrt{\frac{l}{g}} && \text{in variables} \\t &= c\sqrt{\frac{l}{g}}\end{aligned}$$

Exercise 2 *Shock wave.*

$$\begin{aligned}t &\propto \rho^\alpha E^\beta t^\gamma && \text{in variables} \\[L] &= [ML^{-3}]^\alpha [ML^2T^{-2}]^\beta [T]^\gamma && \text{in units}\end{aligned}$$

Dimensional analysis leads to:

$$\begin{aligned}+1\alpha + 1\beta + 0\gamma &= 0 && \text{for mass cancellation} \\-3\alpha + 2\beta + 0\gamma &= 1 && \text{for one length} \\0\alpha - 2\beta + 1\gamma &= 0 && \text{for time cancellation}\end{aligned}$$

Solving this linear system gives $\alpha = -\frac{1}{5}$, $\beta = \frac{1}{5}$, and $\gamma = \frac{2}{5}$.

$$\begin{aligned}[L] &= [ML^{-3}]^{-\frac{1}{5}} [ML^2T^{-2}]^{\frac{1}{5}} [T]^{\frac{2}{5}} && \text{in units} \\t &\propto \rho^{-\frac{1}{5}} E^{\frac{1}{5}} t^{\frac{2}{5}} = \left(\frac{Et^2}{\rho}\right)^{\frac{1}{5}} && \text{in variables} \\t &= k\left(\frac{Et^2}{\rho}\right)^{\frac{1}{5}}\end{aligned}$$

Exercise 3 *Turkey Cooking Times.*

How does cooking time change if we double the weight of a turkey?

Let's assume that Weight \propto Volume, and that a turkey is well approximated by a ball (a solid sphere), then when weight is doubled, volume increases $2^{3/3}$, area increases $2^{2/3}$, and distance to center increases $2^{1/3}$.

Say an 8lb turkey takes 3.5 hours to cook. How long should a 16lb turkey take to cook? Doubling the weight, doubles ($2^{3/3}$) the volume, so it should double the time to cook. But the surface area increases by $2^{2/3}$, allowing more heat transfer, so the time to cook decreases by $2^{-2/3}$. But the heat has further to travel by $2^{1/3}$, increasing the time by $2^{1/3}$. Therefore, the cooking time should increase by $2^{3/3}2^{-2/3}2^{1/3} = 2^{2/3} = 1.587$ times. For a general increase from weight w_1 to weight w_2 , the increase of time will be $\frac{t_2}{t_1} \propto \frac{w_2}{w_1} 2^{2/3}$.

This table of actual recommended¹ cooking times agrees closely with the reasoning above.

Wt (lb)	Time (h)	Considered Increases		Theory Increase
8	3.5	8 to 12 is 1.5	4.5/3.5=1.29	$1.5^{2/3} = 1.31$
12	4.5	8 to 16 is 2	5.5/3.5=1.57	$2^{2/3} = 1.587$
16	5.5	12 to 24 is 2	7.0/4.5=1.56	$2^{2/3} = 1.587$
24	7.0	8 to 24 is 3	7/3.5=2	$3^{2/3} = 2.08$

¹<http://familyinternet.about.com/cs/recipes1/a/blturkey.htm>

Exercise 4 *BMI.*

Height and Weight data for 252 individuals² is used to determine whether Americans scale as $W \propto H^2$, as suggested by the BMI definition of $B \equiv \frac{W}{H^2}$.

Does data support this BMI relationship? Figure 1 shows that the observed relationship between weight and height is

$$\log(W) = 3.1379 + 2.1488 \log(H)$$

with a 95% confidence interval on the slope of (1.7097, 2.5879), which is not different from 2, but with lousy predictive ability with $R^2 = 0.2709$. On average, BMI appears accurate but not precise, that is, it is correctly centered but the variation about the center is large.

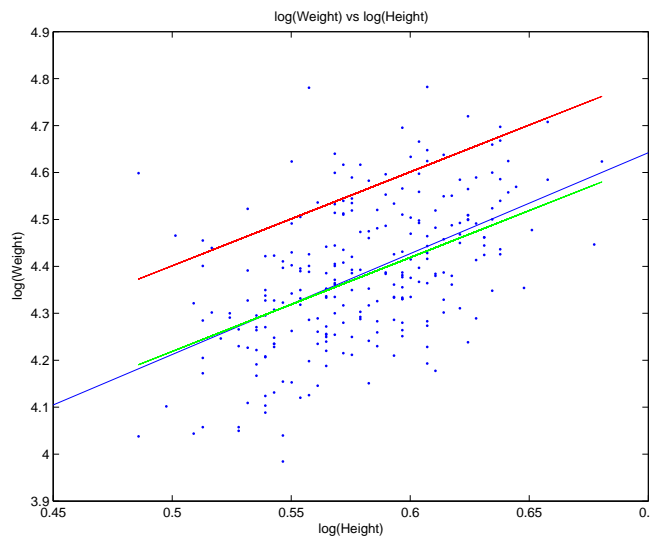


Figure 1: Plots of $\log(\text{Weight})$ vs $\log(\text{Height})$, with lower line of BMI=25 and least-squares line coinciding at separating normal (low) from overweight (mid), and upper line of BMI=30 separating obese (high).

The empirical formula for surface area of a human in terms of height and weight is $A \propto H^{0.725}W^{0.425}$, assuming height and weight are independent (BIG assumption).

Is this consistent with isometry? Let's consider that a person is proportional to a circular cylinder to relate surface area, height, and weight (weight is proportional to volume). With the assumption that height and weight are independent, we see that $A \propto H$ when weight is fixed, and that $A \propto W^{1/2}$ as the radius increases when height is fixed. Therefore, $H \propto A \propto W^{1/2}$, or $W \propto H^2$. Therefore, BMI agrees with isometry and the definition of BMI.

²Roger W. Johnson. Fitting Percentage of Body Fat to Simple Body Measurements. Journal of Statistics Education v.4, n.1 (1996). <http://www.amstat.org/publications/jse/v4n1/datasets.johnson.html>

Exercise 5 *Fluid flow.*

Finding a dimensionless variable. We want to find a product of powered quantities where all the units cancel.

$$1 \propto Q^c P^d \mu^e \rho^f \quad \text{in variables}$$

$$[1] = [L^3 T^{-1}]^c [M T^{-2} L^{-1}]^d [M L^{-1} T^{-1}]^e [M L^{-3}]^f \quad \text{in units}$$

$$\begin{aligned} +3c - 1d - 1e - 3f &= 0 && \text{for length cancellation} \\ \text{Dimensional analysis leads to: } +0c + 1d + 1e + 1f &= 0 && \text{for mass cancellation} \\ -1c - 2d - 1e + 0f &= 0 && \text{for time cancellation} \end{aligned}$$

Solving this linear system requires one constraint, when $f = 1$ then $c = 2/3, d = 1/3, e = -4/3$. Alternatively, $c = 2, d = 1, e = -4, f = 3$, or in general $c = 2f/3, d = f/3, e = -4f/3$. Thus, the dimensionless variable is

$$1 \propto Q^{2f/3} P^{f/3} \mu^{-4f/3} \rho^f \quad \text{in variables}$$

$$[1] = [L^3 T^{-1}]^{2f/3} [M T^{-2} L^{-1}]^{f/3} [M L^{-1} T^{-1}]^{-4f/3} [M L^{-3}]^f \quad \text{in units}$$

$$= L^0 M^0 T^0$$

If $Q = P/R$ then $R = P/Q$:

$$\begin{aligned} R &= P Q^{-1} \\ &= [M T^{-2} L^{-1}] [L^3 T^{-1}]^{-1} \\ &= [M T^{-1} L^{-4}] \end{aligned}$$

I think we want to remove the dependency on μ instead of ρ , and in this case, by cancelling the dimensions in μ of $[M L^{-1} T^{-1}]$ we are left with $[L^{-3}]$, which is precisely the units of $R = c \frac{l}{r^4}$.

Exercise 6 *Koch curve.*

Number of segments 4^n each of length $\frac{1}{3}^n l$, where $n = 0$ indicates a line of length l .

The total length is $L = \frac{4^n}{3} l$.

The fractal dimension D is a function of number of self-similar pieces n and the magnification factor m . The Koch curve creates 4 pieces each of size $\frac{1}{3}$ in each stage, so $n = 4$ and $m = 3$.

$$D = \frac{\log n}{\log m} = \frac{\log 4}{\log 3} = \frac{1.386}{1.099} = 1.262$$

SCALING OF ELEMENTS IN SOLAR SYSTEM AND HUMAN BODY,
LOG(ATOMICMASS) VS LOG(ABUNDANCE)

I'm using available data on atomic elements³ and the abundance in our solar system⁴ and the average human body⁵. It is interesting that the proportions of elements decrease in a power relationship with their atomic weights to the same power, both in the solar system and in the human body, though the relationship is stronger in the solar system.

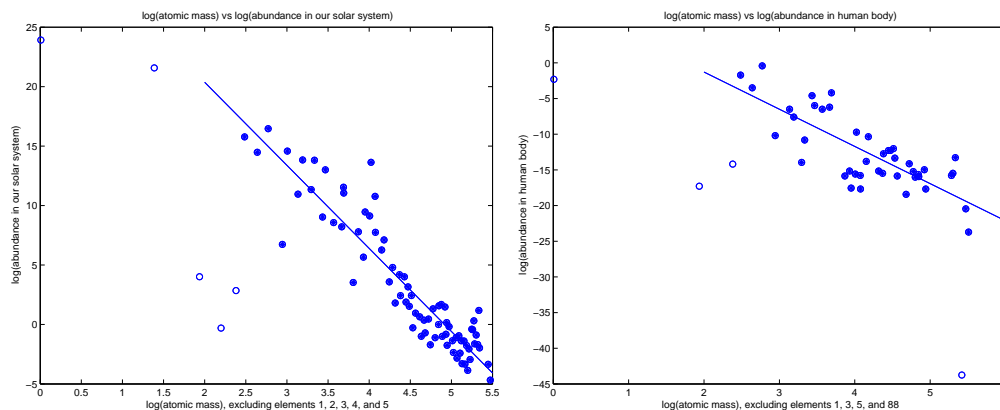


Figure 2: Plots of $\log(\text{AtomicMass})$ vs $\log(\text{Abundance})$, excluding outliers results in steeper slopes. (Left) Solar System indicates a slope of roughly $-1/8$, (Right) Human Body also indicates a slope of roughly $-1/8$.

Figure 2 shows the straight-line relationship between the log of atomic mass versus the log of elemental abundance in the solar system, improved when not considering the five lightest elements. The power scaling between atomic mass, M , and abundance, A , is

$$\log(A) = 4.8540 - 0.1234 \log(M)$$

with a 95% confidence interval on the slope of $(-0.1347, -0.1122)$, which is not different from $-1/8 = 0.125$, and $R^2 = 0.8621$.

Similarly for the log of elemental abundance in the human body, which was also improved slightly when not considering the first three most abundant elements. The power scaling between atomic mass, M , and abundance, A , is

$$\log(A) = 2.6547 - 0.1200 \log(M)$$

with a 95% confidence interval on the slope of $(-0.1493, -0.0907)$, which is not different from $-1/8 = 0.125$, and $R^2 = 0.62542$.

³FYI: With initial investigation, it appears half-life times for radioactive isotopes follow rough power law relationships with the absolute value of the difference between the number of neutrons in the radioactive isotope compared to the stable isotope.

⁴http://en.wikipedia.org/wiki/Cosmochemical_Periodic_Table_of_the_Elements_in_the_Solar_System
K. Lodders, 2003, Solar System Abundances and Condensation Temperatures of the Elements, *Astrophys. J.* 591, 1220-1247. Elements and their abundance measured in number of atoms found per 10^6 atoms of silicon.

⁵http://en.wikipedia.org/wiki/Chemical_makeup_of_the_human_body
Emsley, J. (1998). *The Elements*, 3rd Ed., Clarendon Press:Oxford. The data gives the proportions of elements in the average 70kg human adult.