CS 530 Geometric and Prob. Methods in CS
Homework 6

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Note: All code can be found in the Appendix.

Exercise 1 *Mother wavelet moments.*

$$
\begin{aligned}
M_n\{\Psi\} &= \int_{-\infty}^{\infty} t^n \Psi(t)\,\mathrm{d}t \\
f(t) &= e^{-\pi t^2} \\
f'(t) &= -2\pi t e^{-\pi t^2} \\
f''(t) &= e^{-\pi t^2}(2\pi t^2 - 1)
\end{aligned}
$$

(a) Show $M_0\{f'\} = 0$.

$$
\begin{aligned}
M_0\{f'\} &= \int_{-\infty}^{\infty} t^0(-2\pi t e^{-\pi t^2})\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} -2\pi t e^{-\pi t^2}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} [\text{odd}] \times [\text{even}]\,\mathrm{d}t \\
&= 0
\end{aligned}
$$

(b) Show $M_0\{f''\} = M_1\{f''\} = 0$.

$$
\begin{aligned}
M_0\{f''\} &= \int_{-\infty}^{\infty} t^0(e^{-\pi t^2}(2\pi t^2 - 1))\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{-\pi t^2}(2\pi t^2 - 1)\,\mathrm{d}t \\
&= f'(t)|_{-\infty}^{\infty} \\
&= -2\pi t e^{-\pi t^2}|_{-\infty}^{\infty} \\
&= 0 - 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
M_1\{f''\} &= \int_{-\infty}^{\infty} t^1 (e^{-\pi t^2}(2\pi t^2 - 1))\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} 2\pi t^3 e^{-\pi t^2} - t e^{-\pi t^2}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} 2\pi t^3 e^{-\pi t^2}\,\mathrm{d}t - \int_{-\infty}^{\infty} t e^{-\pi t^2}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} [\text{odd}] \times [\text{even}]\,\mathrm{d}t - \int_{-\infty}^{\infty} [\text{odd}] \times [\text{even}]\,\mathrm{d}t \\
&= 0 - 0 = 0
\end{aligned}
$$

Exercise 2 *Hex-vector Frame.*

The frame $\mathcal{F}$ is

$$
\begin{aligned}
\mathbf{F} &= [f_1, f_2, f_3, f_4, f_5, f_6] \\
&= \frac{1}{2}\begin{bmatrix} 1 & 1 & -2 & -1 & -1 & 2 \\ \sqrt{3} & -\sqrt{3} & 0 & -\sqrt{3} & \sqrt{3} & 0 \end{bmatrix}
\end{aligned}
$$

and shown in Figure 1 on page 2.

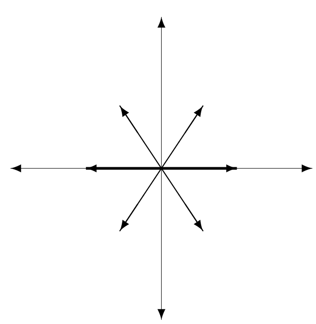

Figure 1: Exercise 2: Frame $\mathcal{F}$.

(a) Two representations. Representations: (1) $[f_1, f_6]$, (2) $[f_4, f_3]$. A representation is given by $[a, b]'$ in

$$\left[\begin{array}{cc} f_i & f_j \end{array}\right] \left[\begin{array}{c} a \\ b \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right] = x$$

and when an inverse exists is solved for by

$$\left[\begin{array}{c} a \\ b \end{array}\right] = \left[\begin{array}{cc} f_i & f_j \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

(1) $[f_1, f_6]$

$$\begin{aligned}
\left[\begin{array}{c} a \\ b \end{array}\right] &= \left[\begin{array}{cc} f_1 & f_6 \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \\
&= \left[\begin{array}{cc} 1 & \frac{1}{2} \\ 0 & \sqrt{\frac{3}{4}} \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \\
&= \left[\begin{array}{cc} 1 & -\sqrt{\frac{1}{3}} \\ 0 & \sqrt{\frac{4}{3}} \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \\
&= \left[\begin{array}{c} 1 - \sqrt{\frac{1}{3}} \\ \sqrt{\frac{4}{3}} \end{array}\right]
\end{aligned}$$

(1) $[f_4, f_3]$

$$\begin{aligned}
\left[\begin{array}{c} a \\ b \end{array}\right] &= \left[\begin{array}{cc} f_4 & f_3 \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \\
&= \left[\begin{array}{cc} -1 & -\frac{1}{2} \\ 0 & -\sqrt{\frac{3}{4}} \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \\
&= -1 \left[\begin{array}{cc} 1 & \frac{1}{2} \\ 0 & \sqrt{\frac{3}{4}} \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \\
&= -1 \left[\begin{array}{cc} 1 & -\sqrt{\frac{1}{3}} \\ 0 & \sqrt{\frac{4}{3}} \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \\
&= -1 \left[\begin{array}{c} 1 - \sqrt{\frac{1}{3}} \\ \sqrt{\frac{4}{3}} \end{array}\right] \\
&= \left[\begin{array}{c} -1 + \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{4}{3}} \end{array}\right]
\end{aligned}$$

(b) Show infinite representations. Consider the frame subset $\mathbf{F}_s = [f_1, f_6, f_3]$, where $f_3 = -f_6$. The representation of a vector $x \in \Re$ is

$$x = \begin{bmatrix} f_1 & f_6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f_1 & f_6 & f_3 \end{bmatrix} \begin{bmatrix} a+k \\ b \\ k \end{bmatrix}$$

where $k \in \Re$. Therefore, there are an infinite number of representations.

(c) $\mathcal{F}$ to standard basis.

$$\mathbf{F}'(\mathbf{F}\mathbf{F}')^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 1 & -2 & -1 & -1 & 2 \\ \sqrt{3} & -\sqrt{3} & 0 & -\sqrt{3} & \sqrt{3} & 0 \end{bmatrix}'$$

(d) standard basis to $\mathcal{F}$.

$$(\mathbf{F}\mathbf{F}')\mathbf{F} = \frac{3}{2} \begin{bmatrix} 1 & 1 & -2 & -1 & -1 & 2 \\ \sqrt{3} & -\sqrt{3} & 0 & -\sqrt{3} & \sqrt{3} & 0 \end{bmatrix}$$

In both (c) and (d) just divide or multiply by the multiplicity.

Exercise 3 *Two channel subband coding.*

The plot in Figure 2 on page 5 shows the row for the original signal. The signal is in Figure 4 on page 7.

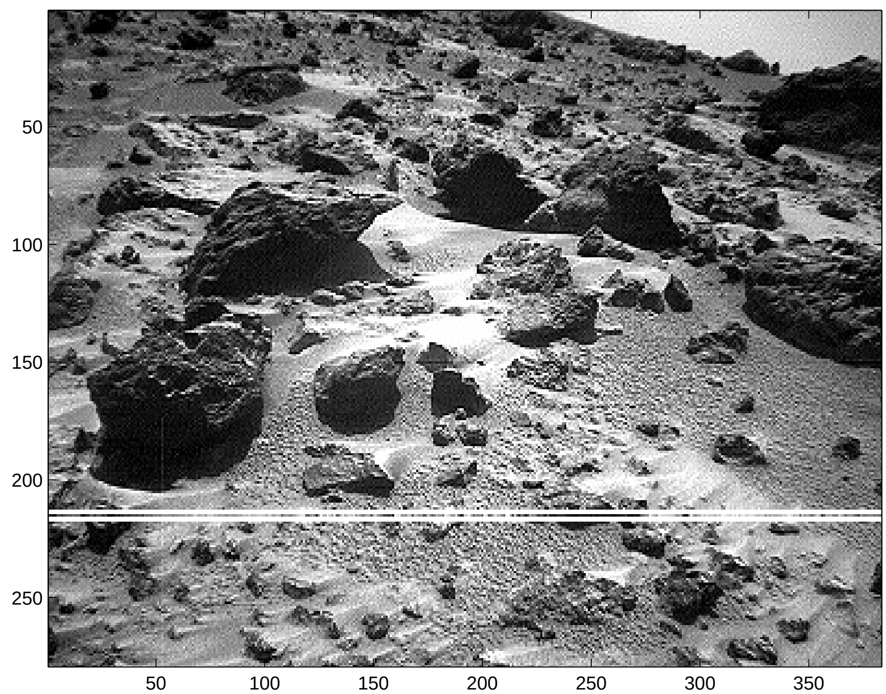

Figure 2: Exercise 3: Row 215 from mars.pgm image is highlighted between white rows.

(a,b) Lowpass and Highpass filters. The plot in Figure 3 on page 6 show the lowpass and highpass filters.

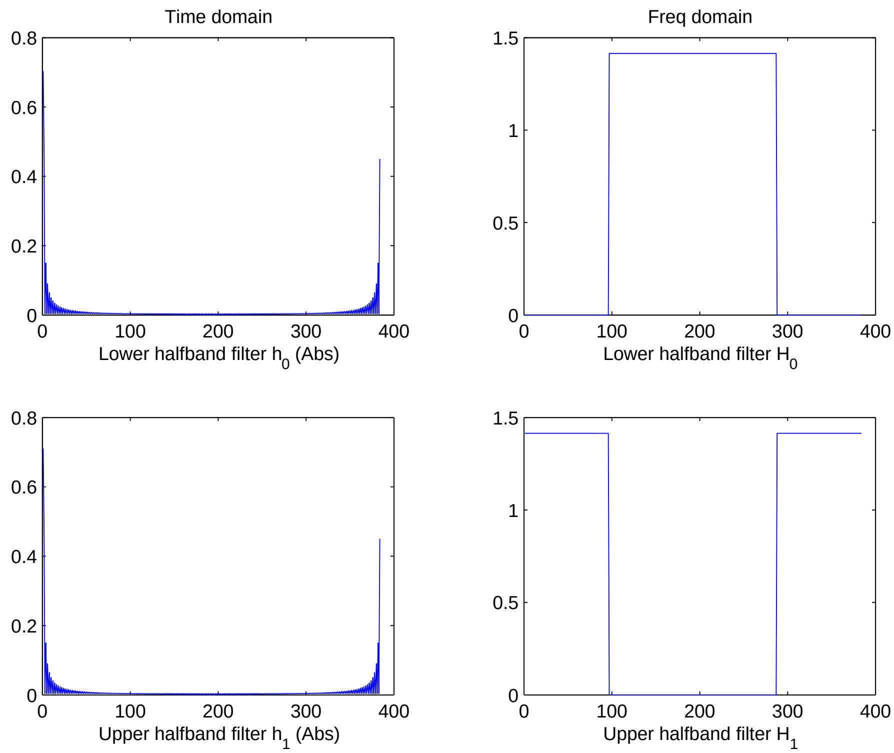

Figure 3: Exercise 3a,b: Lowpass and Highpass filters.

(c,d) Time and Frequency domain plots at every point in two channel subband coding process. The plots in Figure 4 on page 7 through Figure 13 on page 12 show the requested plots.

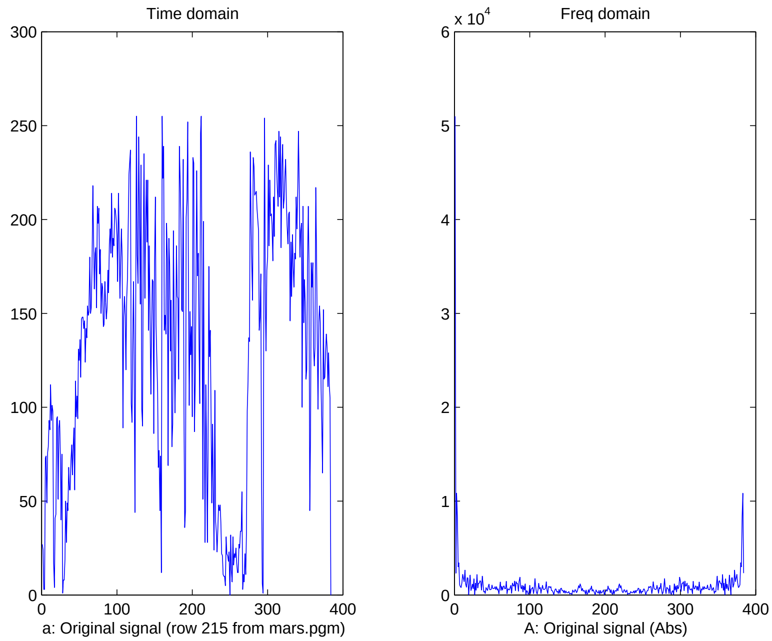

Figure 4: Exercise 3c,d: A, Original signal.

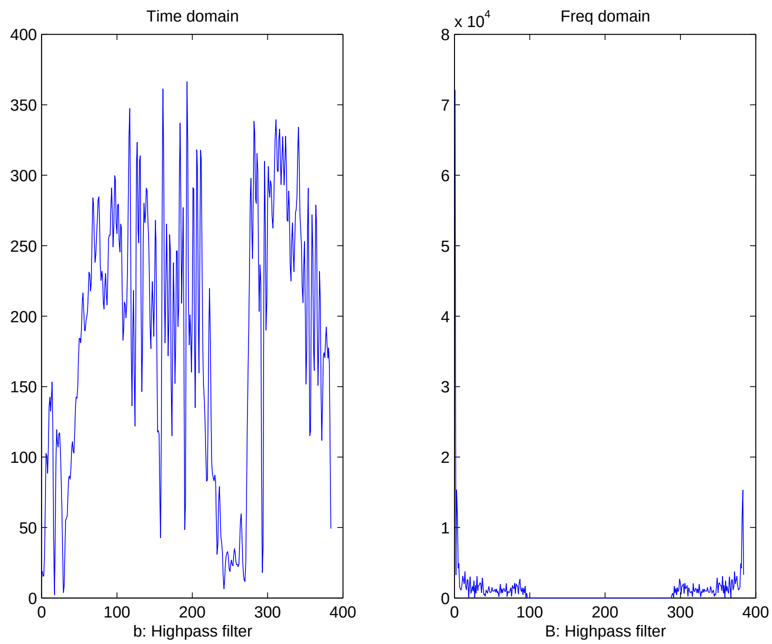

Figure 5: Exercise 3c,d: B, Highpass filter.

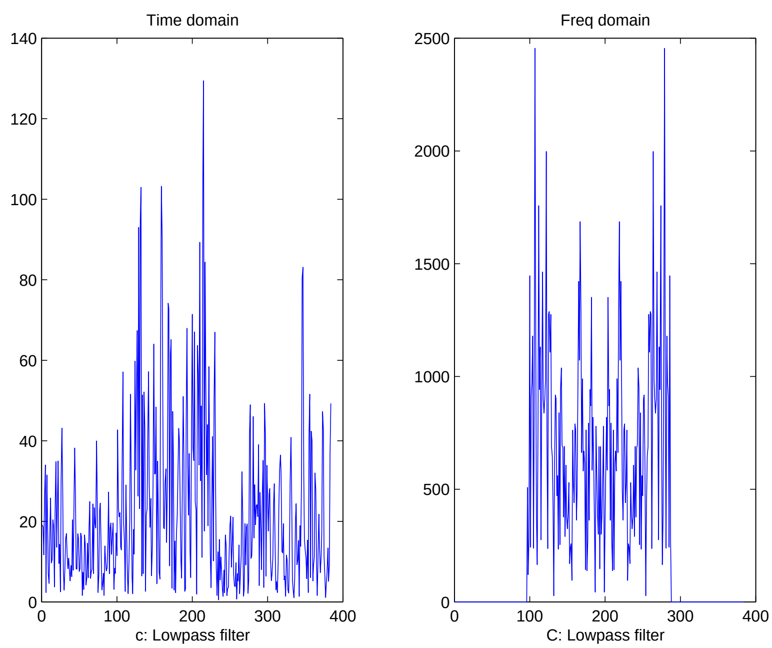

Figure 6: Exercise 3c,d: C, Lowpass filter.

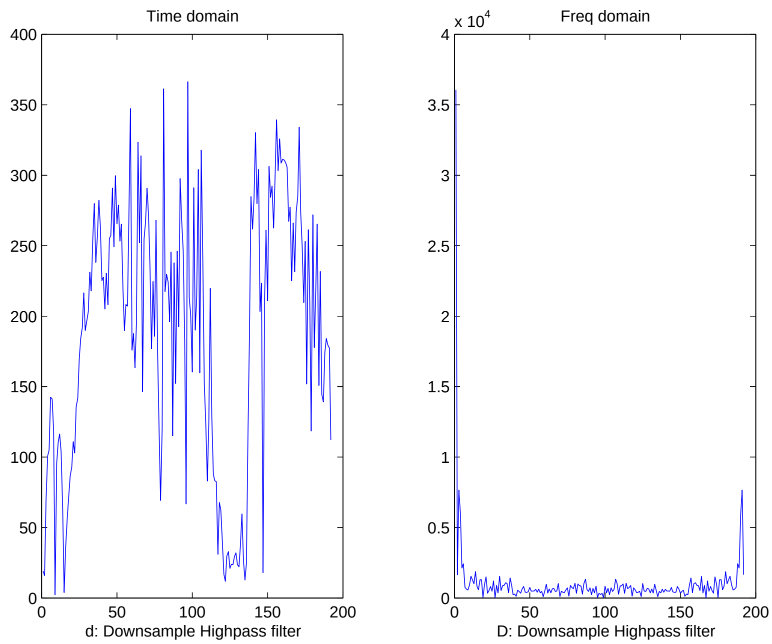

Figure 7: Exercise 3c,d: D, Downsample from highpass.

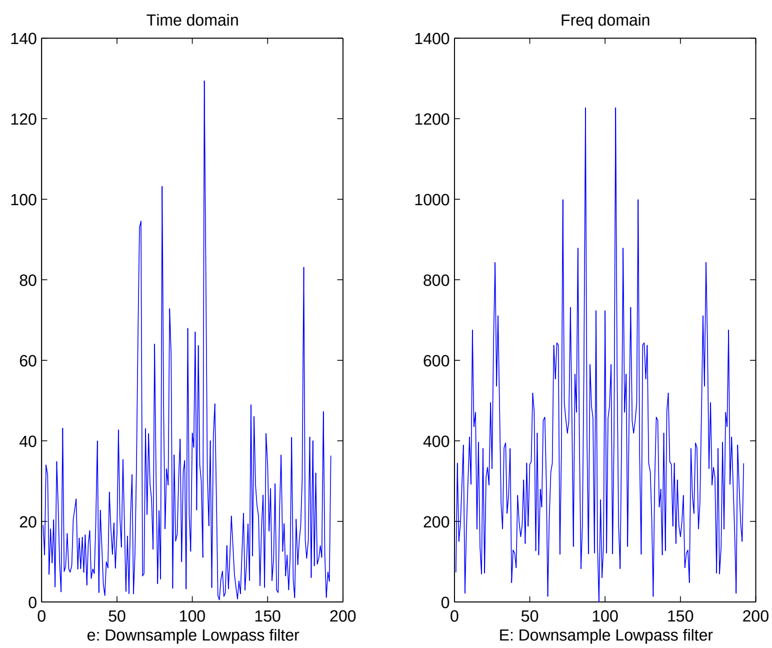

Figure 8: Exercise 3c,d: E, Downsample from lowpass.

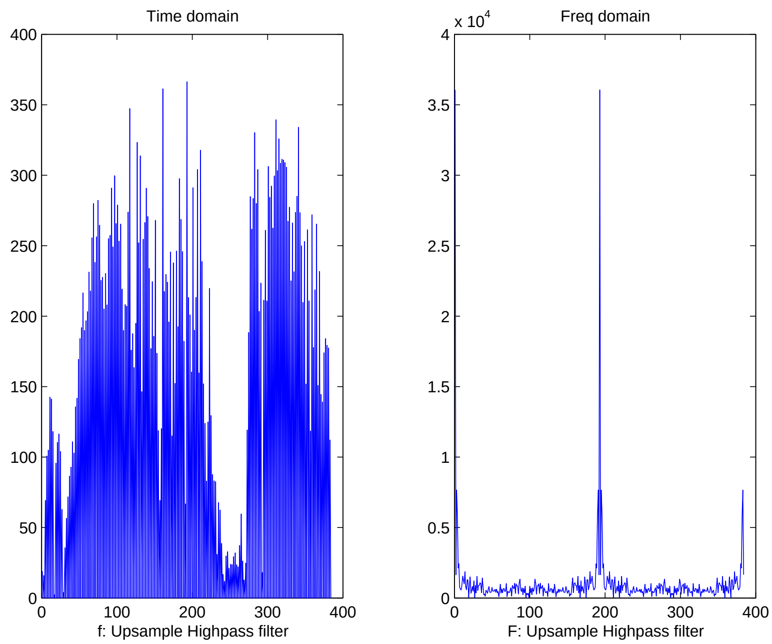

Figure 9: Exercise 3c,d: F, Upsample from highpass.

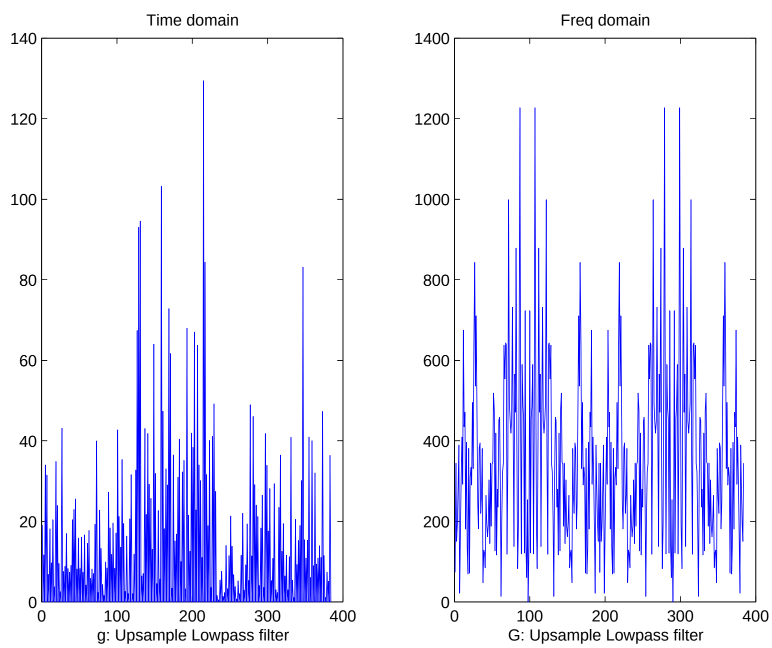

Figure 10: Exercise 3c,d: G, Upsample from lowpass.

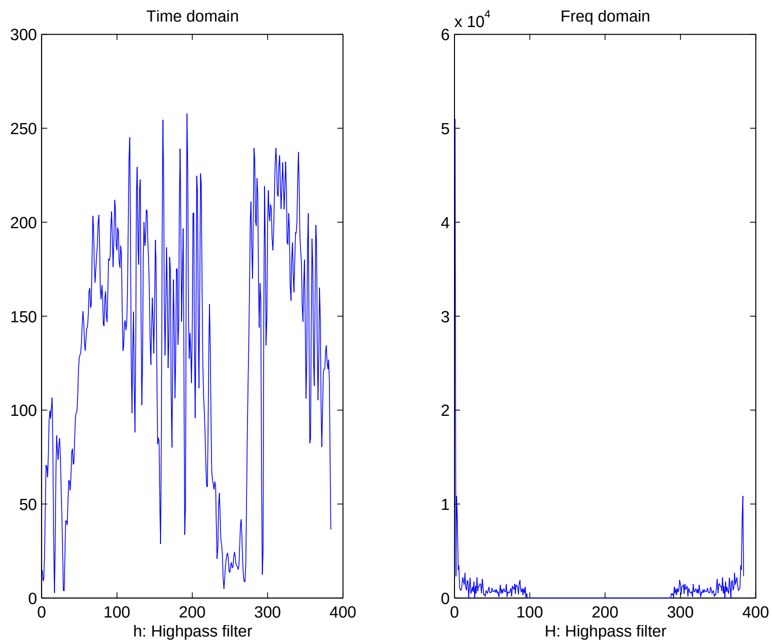

Figure 11: Exercise 3c,d: H, Highpass filter.

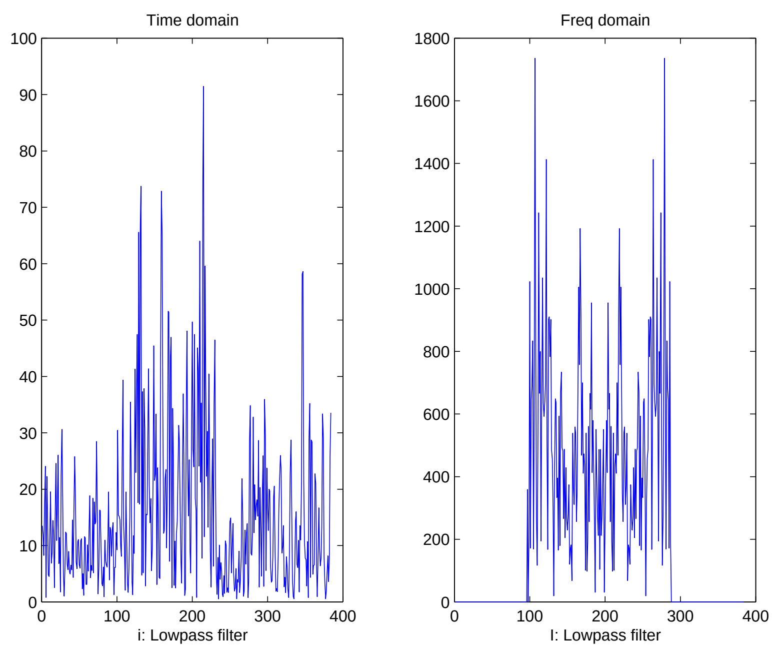

Figure 12: Exercise 3c,d: I, Lowpass filter.

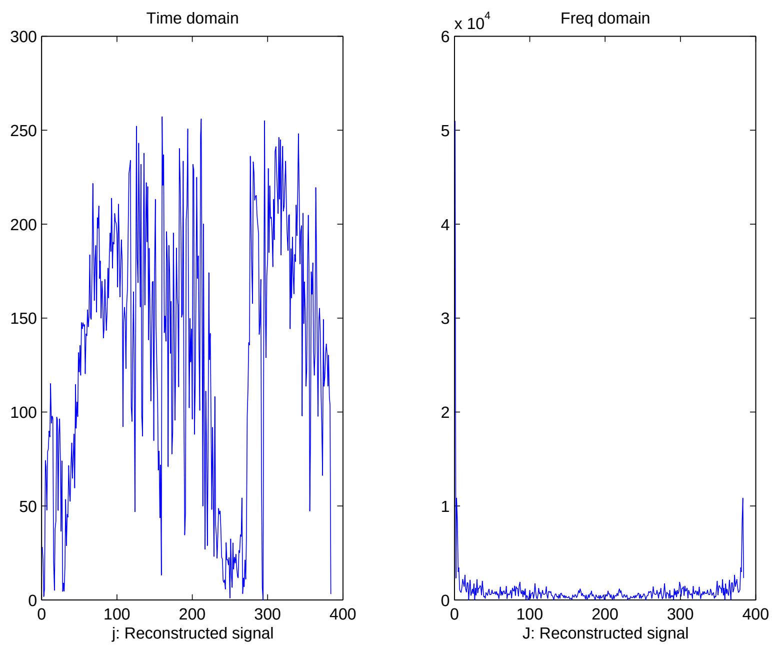

Figure 13: Exercise 3c,d: J, Recombine signals.

Appendix

Matlab code used for the exercises

```
clear all
format compact
format short
%%% Exercise 3 ================================================================
% Read in ppm image data
fn='mars.pgm';  [mars]=readpgm(fn); % Gray
white=max(max(mars));
mars([213:214 216:217],:) = white*ones(4,383);
gdisplay(mars);                        % see mars
plot_name = strcat('gpcshw06-3.eps'); print(gcf, '-depsc2', plot_name);      % print plot
x = [mars(215,:) 0]';                   % row 215 of mars image, make even length
N=length(x);                           % length N=384

X = fft(x);  % frequency domain

% 3a ====================
% Low and High filters Freq domain
H0 = zeros(N,1);
H1 = zeros(N,1);
for i=1:N
   if or(i<=(N/4),i>(3*N/4-1));
      H1(i) = 1;
   else;
      H0(i) = 1;
   end;
end
H0=H0*sqrt(2);
H1=H1*sqrt(2);

% 3b ====================
% Low and High filters Time domain
h0 = ifft(H0);
h1 = ifft(H1);

clf;
subplot(2,2,1); plot(abs(h0)); xlabel('Lower halfband filter h_0 (Abs)');  title('Time domain')
subplot(2,2,2); plot(1:N,H0);  xlabel('Lower halfband filter H_0');        title('Freq domain');
subplot(2,2,3); plot(abs(h1)); xlabel('Upper halfband filter h_1 (Abs)');  %title('Time domain')
subplot(2,2,4); plot(1:N,H1);  xlabel('Upper halfband filter H_1');        %title('Freq domain');
plot_name = strcat('gpcshw06-3ab.eps'); print(gcf, '-depsc2', plot_name);      % print plot

% 3cd ====================
%% A %%%%%%%%%%%%%%%%%%%%%
% Original signal Time domain
a=x;
% FFT of signal Freq domain
A=X;
clf;
subplot(1,2,1); plot(abs(a)); xlabel('a: Original signal (row 215 from mars.pgm)'); title('Time domain');
subplot(1,2,2); plot(abs(A)); xlabel('A: Original signal (Abs)'); title('Freq domain')
plot_name = strcat('gpcshw06-3cA.eps'); print(gcf, '-depsc2', plot_name);      % print plot

%% B %%%%%%%%%%%%%%%%%%%%%
% Highpass filter
B=A.*H1;
b=ifft(B);
clf;
subplot(1,2,1); plot(abs(b)); xlabel('b: Highpass filter'); title('Time domain');
subplot(1,2,2); plot(abs(B)); xlabel('B: Highpass filter'); title('Freq domain')
plot_name = strcat('gpcshw06-3cB.eps'); print(gcf, '-depsc2', plot_name);      % print plot

%% C %%%%%%%%%%%%%%%%%%%%%
% Lowpass filter
C=A.*H0;
c=ifft(C);
clf;
subplot(1,2,1); plot(abs(c)); xlabel('c: Lowpass filter'); title('Time domain');
subplot(1,2,2); plot(abs(C)); xlabel('C: Lowpass filter'); title('Freq domain')
plot_name = strcat('gpcshw06-3cC.eps'); print(gcf, '-depsc2', plot_name);      % print plot

% downsample filter
downsample = repmat([1;0],N/2,1);  % [1 0 1 0 ... 1 0]' % zero even time values

%% D %%%%%%%%%%%%%%%%%%%%%
% Downsample from highpass
d = b(find(downsample));
D = fft(d);
clf;
subplot(1,2,1); plot(abs(d)); xlabel('d: Downsample Highpass filter'); title('Time domain');
subplot(1,2,2); plot(abs(D)); xlabel('D: Downsample Highpass filter'); title('Freq domain')
plot_name = strcat('gpcshw06-3cD.eps'); print(gcf, '-depsc2', plot_name);      % print plot
```

```
%% E %%%%%%%%%%%%%%%%%%%%%
% Downsample from lowpass
e = c(find(downsample));
E = fft(e);
clf;
subplot(1,2,1); plot(abs(e)); xlabel('e: Downsample Lowpass filter'); title('Time domain');
subplot(1,2,2); plot(abs(E)); xlabel('E: Downsample Lowpass filter'); title('Freq domain')
plot_name = strcat('gpcshw06-3cE.eps'); print(gcf, '-depsc2', plot_name);      % print plot

%% F %%%%%%%%%%%%%%%%%%%%%
% Upsample from highpass
f = reshape([d zeros(size(d))]',N,1);   % inserting 0s every other place
F = fft(f);
clf;
subplot(1,2,1); plot(abs(f)); xlabel('f: Upsample Highpass filter'); title('Time domain');
subplot(1,2,2); plot(abs(F)); xlabel('F: Upsample Highpass filter'); title('Freq domain')
plot_name = strcat('gpcshw06-3cF.eps'); print(gcf, '-depsc2', plot_name);      % print plot

%% G %%%%%%%%%%%%%%%%%%%%%
% Upsample from lowpass
g = reshape([e zeros(size(e))]',N,1);   % inserting 0s every other place
G = fft(g);
clf;
subplot(1,2,1); plot(abs(g)); xlabel('g: Upsample Lowpass filter'); title('Time domain');
subplot(1,2,2); plot(abs(G)); xlabel('G: Upsample Lowpass filter'); title('Freq domain')
plot_name = strcat('gpcshw06-3cG.eps'); print(gcf, '-depsc2', plot_name);      % print plot

%% H %%%%%%%%%%%%%%%%%%%%%
% Highpass filter
H=F.*H1;
h=ifft(H);
clf;
subplot(1,2,1); plot(abs(h)); xlabel('h: Highpass filter'); title('Time domain');
subplot(1,2,2); plot(abs(H)); xlabel('H: Highpass filter'); title('Freq domain')
plot_name = strcat('gpcshw06-3cH.eps'); print(gcf, '-depsc2', plot_name);      % print plot

%% I %%%%%%%%%%%%%%%%%%%%%
% Lowpass filter
I=G.*H0;
i=ifft(I);
clf;
subplot(1,2,1); plot(abs(i)); xlabel('i: Lowpass filter'); title('Time domain');
subplot(1,2,2); plot(abs(I)); xlabel('I: Lowpass filter'); title('Freq domain')
plot_name = strcat('gpcshw06-3cI.eps'); print(gcf, '-depsc2', plot_name);      % print plot

%% J %%%%%%%%%%%%%%%%%%%%%
% Recombine signals
j=h+i;
J=H+I;
clf;
subplot(1,2,1); plot(abs(j)); xlabel('j: Reconstructed signal'); title('Time domain');
subplot(1,2,2); plot(abs(J)); xlabel('J: Reconstructed signal'); title('Freq domain')
plot_name = strcat('gpcshw06-3cJ.eps'); print(gcf, '-depsc2', plot_name);      % print plot
```