A Better Paper Helicopter: Process Optimization Using Response Surface Methodology

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Abstract

Response surface methodology (RSM) is a collection of statistical and mathematical techniques used to parsimoniously explore and improve a process. The major tools used are (1) design of experiments (DOE), (2) multiple regression, and (3) optimization. No current research is discussed in this talk. Instead, the intention is to provide an introduction to this powerful collection of methods, applicable to any process with controllable inputs. Some potential objectives and applications include mapping the response surface over a region of interest, optimizing the response (minimize, maximize, or target), or achieving a set of specifications and/or requirements. Techniques are introduced and then illustrated with the application of maximizing the flight time of a paper helicopter. Helicopters will be folded and dropped.

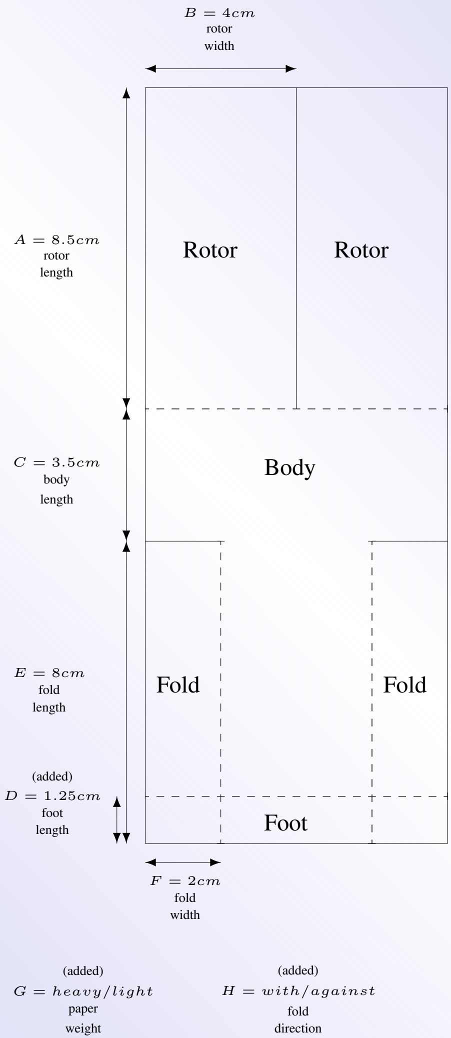


Figure 1: Initial Helicopter Pattern

Outline

- Introduce the ideas of response surface methodology
- Curve estimation with first- and second-order polynomials
- Sequential process of improvement and optimization
- Regression and factorial designs are discussed
- Experiments that lead us to a winning helicopter

1. Response Surface Methodology

Response surface methodology (RSM) is a collection of statistical and mathematical techniques used to parsimoniously explore and improve a process [6].

The major tools used are

- Design of Experiments (DOE)
- Multiple regression
- Optimization

Our **goal** is to find the settings of factors that maximize flight time.

Model:

- Response (possibly multivariate), y
- Factor inputs, $\xi_1, \ldots, \xi_k$
- $y = f(\xi_1, \ldots, \xi_k) + \varepsilon$, where f is an unknown function (usually assumed smooth)
- ε is random error with expected value $\mathrm{E}(\varepsilon) = 0$ and common variance $\mathrm{Var}(\varepsilon) = \sigma^2$

The **response surface** is the expected flight time for varying sets of factor inputs defined by $\mathrm{E}(y) = f(\xi_1, \ldots, \xi_k)$.

For convenience, the natural variables, $\xi_1, \ldots, \xi_k$, in their original units are often transformed to unitless coded variables, $x_1, \ldots, x_k$ with zero mean $\mu_i = 0$ and common standard deviation. The response surface used is in terms of the coded variables $\mathrm{E}(y) = f(x_1, \ldots, x_k)$.

Because we often don't know the form of f, we need to approximate it, at least locally.

Taylor's theorem says surfaces look flat up close.

First- and second-order polynomials have the form

$$\begin{aligned} \mathrm{E}(y) \doteq\ & \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{ij} x_i x_j \\ & + \sum_{i=1}^{k} \beta_{ii} x_{ii}^2 . \end{aligned} \tag{1}$$

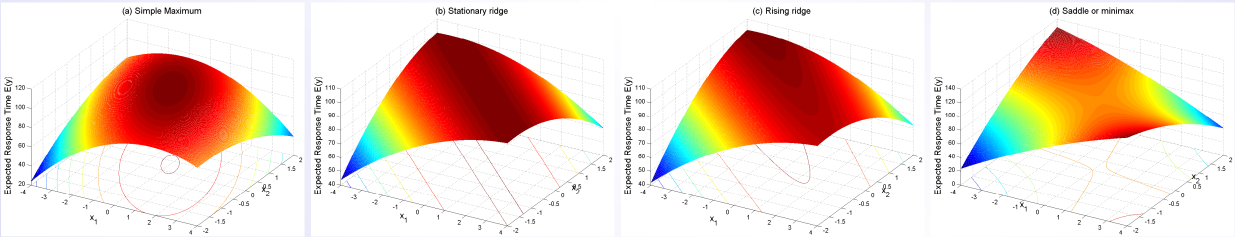


Figure 2: Examples of four common types of response surfaces defined by the second-order model in $k = 2$ variables. (a) Simple maximum has a unique peak at a point, (b) stationary ridge has a common maximum along a line (hyperplane), (c) rising ridge does not have a maximum, but a single direction which most steeply increases, and a (d) saddle has a point which is the maximum in one direction, but a minimum in another direction, hence the name, minimax.

1.1. Sequential Process

The sequential process of RSM has three primary phases:

- PHASE 0: **Brainstorming session** reveals many potential factors influencing the response.

  Screening experiment eliminates many of the trivial variables, retaining the vital few important factors influencing the response.

  The dimension of the experimental region is often greatly reduced, exponentially decreasing the number of further experiments required.

- PHASE 1: Experiment performed to see whether the settings of the important factors produce a near-optimal response.

  If not near optimum, use first-order model to move in the direction of **steepest ascent**.

- PHASE 2: a near-optimum location has been found and a second-order model is used to exhibit the curvature near the optimum in order to **determine optimal conditions**.

  Confirmatory experiment at the optimum location verifies the second-order prediction.

1.2. Multiple Linear Regression Models

Linear models, multiple linear regression models in particular, are used to approximate the response surface.

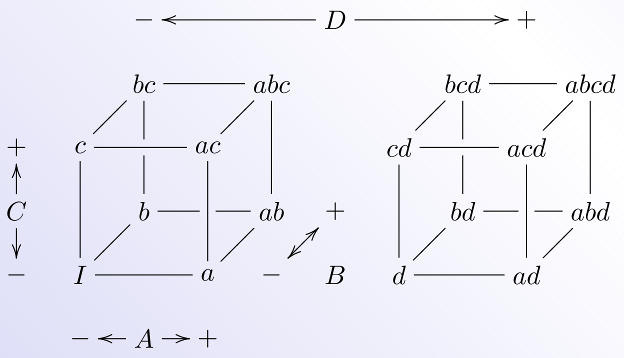
These models are linear with respect to the model coefficients, $\beta_1, \ldots, \beta_k$, but not necessarily in the variables, $x_i, x_i^2, \log(x_i)$.

We will consider first- and second-order models as in equation (1).

2. Factorial Designs

Factorial designs are efficient designs for assessing the marginal and joint effects of several factors on a response.

A 2^k design is a factorial design for k factors, each at two levels $(-1, +1)$.



Factorial designed are used in

- screening experiment to reduce the number of factors to a manageable number
- fitting first-order models to determine the path of steepest ascent
- basic building block for other RSM designs

Center points

- runs at level 0 of all factors
- an inexpensive way of checking curvature
- obtaining an independent estimate of the error variance
- doesn't effect the orthogonality of the design

2.1. Fractional Factorial Designs

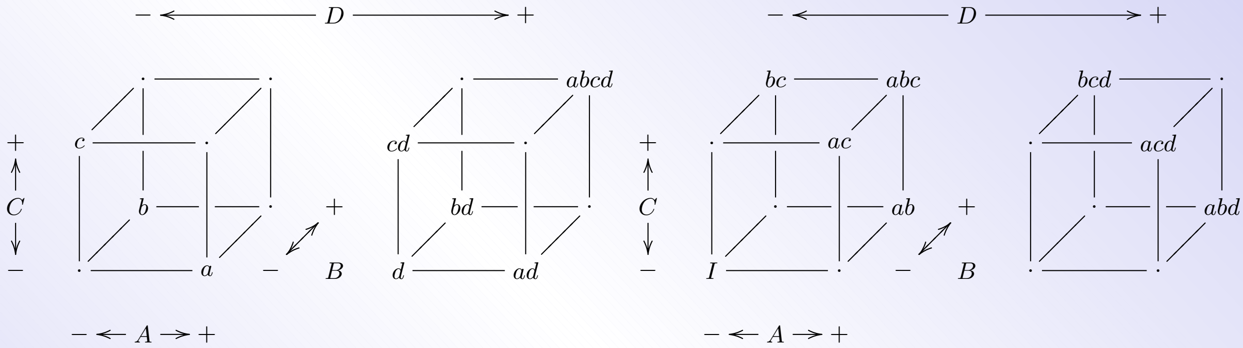
When the number of factors k is large, a 2^k design will have a large number of runs.

For example, $k = 8$ requires $2^8 = 256$ runs, having 255 degrees-of-freedom (df) to estimate factor effects.

- 8 df are used to estimate main effects
- 28 df are used to estimate two-factor interactions
- remaining 219 df are used to estimate three-factor and higher interactions

If reasonable to assume the effect of higher-order interactions is negligible, then main- and low-order effects can be estimated by running a fraction of the number of experiments, for example 2^{k-p}.

Two one-half fractions of the 2^4 design, 2^{4-1} where $I = ABCD$:



Successful use of FFDs relies on three key ideas:

Sparcity of effects principle only a few main effects and/or low-order interactions account for almost all the variation in the response, making the higher-order effects negligible.

Projection property FFDs can be projected into stronger designs, allowing analysis as a full factorial in some factors by ignoring others.

Sequential experimentation allows combinations of FFDs to sequentially assemble a larger design, allowing additional experimental trials to be added to the existing set of trials.

The price paid for the efficiency of the 2^{k-p} design is the ambiguity introduced through an aliasing structure which confounds low-order with high-order effects.

For example, with eight factors, there are $2^8 = 256$ effects: I is the identity; $a, b, \ldots, h$ are main effects; $ab, ac, \ldots, gh$ are two-factor interactions; up to the eight-factor interaction, $abcdefgh$.

In our 2^{8-4} design used in our screening experiment, the aliasing structure based on $E = BCD, F = ACD, G = ABD, H = ABC$ confounds the three-factor and higher interactions with the main effects.

To resolve this ambiguity we rely on philosophical principles and practical methods:

Pareto principle there are vital few important effects and a trivial many.

Hierarchy of importance lower-order effects are more important than higher-order effects.

Knowledge of confounding patterns or aliasing structure and reasoning says that if ab is aliased with ch, and three effects a, b, and $(ab = ch)$ are all judged important, then it is likely that ab is important and ch is not, since main effects a and b were each important.

Conducting further experiments can resolve ambiguities.

2.2. Blocking

Device for reducing variation when all the runs of an experiment can't be performed under homogeneous conditions.

Use the motto, "block what you can, randomize the rest."

Analysis includes a block effect and uses $b - 1$ degrees-of-freedom for b blocks.

Helicopter creation:

(1) paper stock (2) who marks the pattern on the paper (3) who cuts the paper (4) who folds the paper

Helicopter dropping:

(1) conditions of the day (atmospheric pressure, temperature, dew point) (2) conditions of the time (whether processes such as climate control are running, movement by people, or other events effecting air circulation) (3) drop location (4) who drops the helicopters (5) dropping method (6) who records the time

3. Let the contest begin!

3.1. Phase 0: Brainstorm and Screening Experiment

Brainstorm

First a brainstorm about helicopter design: Bring your knowledge and experience forward about helicopter flight — make a few and drop them, try modifications. This is fun!

Factors	Coded Values			Range
	-1	0	1	
A = Rotor Length	5.5	8.5	11.5	6
B = Rotor Width	3	4	5	2
C = Body Length	1.5	3.5	5.5	4
D = Foot Length	0	1.25	2.5	2.5
E = Fold Length	5	8	11	6
F = Fold Width	1.5	2	2.5	1
G = Paper Weight	light	(none)	heavy	(none)
H = Fold Direction	against	(none)	with	(none)

No issue regarding the mixture of continuous (lengths and widths) and categorical (weight and direction) factor levels.

Screening experiment using a 2^{8-4} fractional factorial design with the initial helicopter pattern as our center point.

Screening Experiment

The goal during the screening experiment is to identify those factors with the most influence on the response and then eliminate the remaining factors from further analysis.

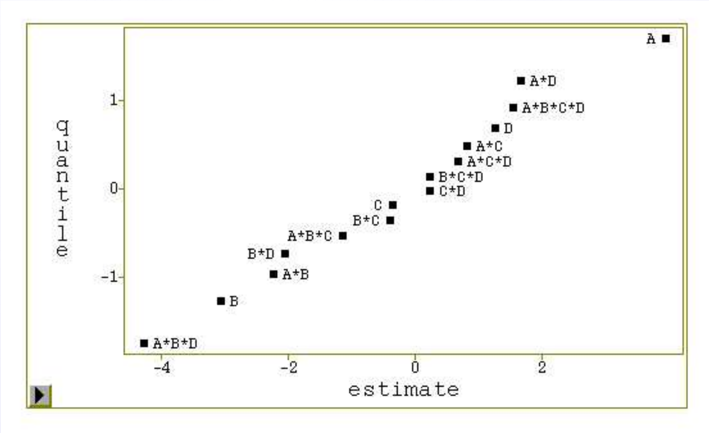
		Factors and Coded Levels								
Obs	Ord	A	B	C	D	E	F	G	H	Time
1	12	−1	−1	−1	−1	−1	−1	−1	−1	11.80
2	7	−1	−1	−1	1	1	1	1	−1	8.29
3	11	−1	−1	1	−1	1	1	−1	1	9.00
4	15	−1	−1	1	1	−1	−1	1	1	7.21
5	1	−1	1	−1	−1	1	−1	1	1	6.65
6	4	−1	1	−1	1	−1	1	−1	1	10.26
7	16	−1	1	1	−1	−1	1	1	−1	7.98
8	8	−1	1	1	1	1	−1	−1	−1	8.06
9	3	1	−1	−1	−1	−1	1	1	1	9.20
10	10	1	−1	−1	1	1	−1	−1	1	19.35
11	9	1	−1	1	−1	1	−1	1	−1	12.08
12	5	1	−1	1	1	−1	1	−1	−1	20.50
13	6	1	1	−1	−1	1	1	−1	−1	13.58
14	13	1	1	−1	1	−1	−1	1	−1	7.47
15	14	1	1	1	−1	−1	−1	−1	1	9.79
16	2	1	1	1	1	1	1	1	1	9.20

Single replicate so **no estimate of error**, since all of the degrees-of-freedom are being used for estimating the factor effects.

Can't use t-tests for testing the significance of factor effects.

A large effect is an outlier, so how would you identify outliers? A graphical method uses a normal quantile plot of the effects with outliers identified as significant effects [2].

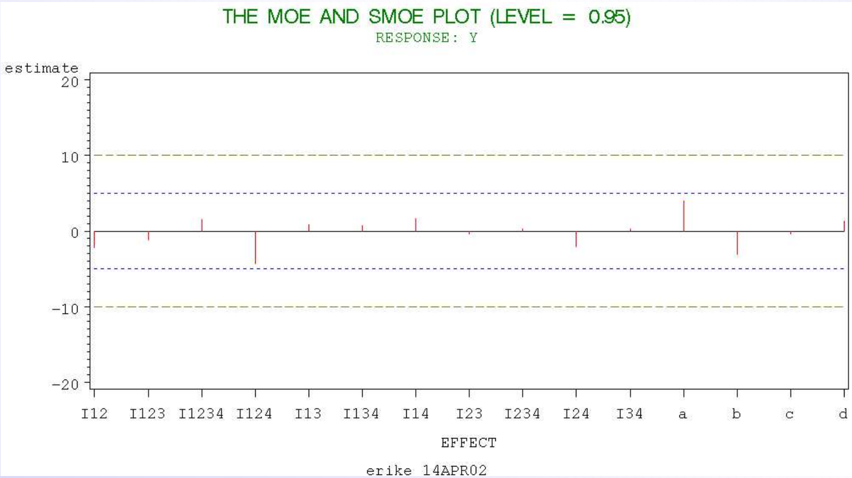
A (rotor length), B (rotor width), and $G = ABD$ (paper weight) are the most likely candidates.



Nongraphical methods include Lenth's (S)MOE methods for unreplicated factorials [4].

Lenth's method calculates MOE (margin-of-error) and SMOE (simultaneous margin-of-error) bars based on the median (not the mean) of m effects.

Inner MOE bars (± 4.94516) or outer SMOE bars (± 10.0394).



(S)MOE effects plot

Label	Factor	Estimate	
A	A	3.9900	*
B	B	−3.0550	*
C	C	−0.3475	
D	D	1.2825	
I234	$E = BCD$	0.2500	
I134	$F = ACD$	0.7000	
I124	$G = ABD$	−4.2825	*
I123	$H = ABC$	−1.1375	
I12	AB	−2.2175	
I13	AC	0.8400	
I14	AD	1.6850	
I23	BC	−0.3850	
I24	BD	−2.0350	
I34	CD	0.2475	
I1234	$ABCD$	1.5625	

Lenth's

No significant factors, even when the response times varied from 7 to 20 seconds?!

Do not be discouraged. We are in control of our own experiment and additional trials are easy. We know having a larger sample reduces estimation error, so. . . one way to reduce the size of those (S)MOE error bars is to run replicated experiments.

Design augmentation with center points provides an estimate of experimental (pure) error (PE), allowing t-tests.

$$\text{(S)MOE} = s_{\text{PE}} \times t_{c-1,\delta}$$

where $\delta = \frac{1+L}{2}$ for MOE and $\delta = \frac{1+L^{1/m}}{2}$ for SMOE, and confidence level $L = 0.95$, and

$$s_{\text{PE}} = \sqrt{\frac{\sum_{i=1}^{n_1}(x_{i1} - \bar{x}_1)^2 + \sum_{i=1}^{n_2}(x_{i2} - \bar{x}_2)^2}{(n_1 - 1) + (n_2 - 1)}} \qquad (2)$$

is the weighted sample standard deviation of the two sets of $n_1 = n_2 = 3$ responses at the $c = 6$ center points.

Center points, and parsimoniously run H only its low level, and run G and both its levels in order to average on either side of its undefined center level.

		Factors and Coded Levels								
Obs	Ord	A	B	C	D	E	F	G	H	Time
1	5	0	0	0	0	0	0	-1	-1	10.52
2	1	0	0	0	0	0	0	-1	-1	10.81
3	3	0	0	0	0	0	0	-1	-1	10.89
4	2	0	0	0	0	0	0	1	-1	15.91
5	4	0	0	0	0	0	0	1	-1	16.08
6	6	0	0	0	0	0	0	1	-1	13.88

Table 1: Screening Drop Results of Center Points

$s_{PE} = 0.8764$, $\text{MOE}_{0.05} = 2.4332$.

Main effects A, B, and $G = ABD$ significant.

We have three main effects, but clearly $G = -1$ (the light paper) gives a higher response. Also, we fix $H = -1$ (fold against) since this setting both gave observably more stable helicopters and tends to have better performance.

With the reduction of factors through lack of significance (C, D, E, F and H) and confinement (G and H), we have only two factors to consider (A and B). Hereafter, we can use the full 2^2 factorial replicated design to start our optimization process.

What's the shortest path to the top of a hill? Always take the steepest possible step.

Approximate method: Make the first step the steepest possible and continue in that direction until two steps head downward.

Then we look around (experiment) at our local maximum and reevaluate our course.

The direction and magnitude of steepest ascent is the **gradient**.

For the response surface $\mathrm{E}(y) = f(x_1, \ldots, x_k)$, the direction of steepest ascent as the gradient is $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_k}\right)'$, a vector of slopes.

In particular, for the first-order model $\mathrm{E}(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$, the path of steepest ascent is in the direction $(\beta_1, \beta_2, \ldots, \beta_k)'$.

Fitting the model $\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \cdots + b_k x_k$, the resulting estimated direction of steepest ascent is $(b_1, b_2, \ldots, b_k)'$.

Obs	A	B	Time
1	-1	-1	10.24
2	-1	-1	9.11
3	1	-1	16.52
4	1	-1	16.99
5	-1	1	10.20
6	-1	1	9.26
7	1	1	10.02
8	1	1	9.94
9	-1	-1	11.31
10	-1	-1	10.94
11	1	-1	12.58
12	1	-1	13.86
13	-1	1	8.20
14	-1	1	9.92
15	1	1	9.95
16	1	1	9.93
17	0	0	11.67
18	0	0	10.74
19	0	0	9.83

Table 2: 2^2 replicated design results are used to determine the path of steepest ascent.

How can we use the three center runs in our experiment to check surface curvature?

If the surface is flat, then the mean response at the center point ($\bar{y}_C$) and the mean of the responses at the corner factorial points ($\bar{y}_F$) will be at the same place.

An estimate of the curvature is the difference $\bar{y}_F - \bar{y}_C$, and if this is large relative to experimental error, there is evidence that the response surface is curved. A simple two-sample t-test does the trick.

The estimated standard error of $\bar{y}_F - \bar{y}_C$ is

$$\hat{\sigma}_{\bar{y}_F - \bar{y}_C} = \sqrt{s^2_{\text{PE}}\left(\frac{1}{n_F} + \frac{1}{n_C}\right)} = \sqrt{0.8464\left(\frac{1}{16} + \frac{1}{3}\right)} = 0.5788 \qquad (3)$$

and the critical cutoff at the 95% level ($L = 0.95$) is

$$\hat{\sigma}_{\bar{y}_F - \bar{y}_C} t_{c-1,(1+L)/2} = 0.5788(2.92) = 1.69 > |\bar{y}_F - \bar{y}_C| = 0.4389. \qquad (4)$$

The difference is not larger than expected so there is insufficient evidence to conclude that there is surface curvature. We prepare to climb the flat face of the hill.

The estimated response surface is

$$\begin{aligned}\hat{y} &= b_0 + b_1 x_1 + b_2 x_2 \\ &= 11.1163 + 1.2881 x_1 - 1.5081 x_2. \end{aligned} \tag{5}$$

A pilot study in the insignificant factors helps to stabilize the helicopters.

We decide favorable levels to be body length $C = 2$, foot length $D = 2$, fold length $E = 6$, and fold width $F = 2$, in their natural units.

Determine step size: $1cm$ in factor A, corresponding to $\frac{1}{3}$ design units.

The corresponding step size in design variable B is then

$$(b_2/b_1)(1/3) = (-1.5081/1.2881)(1/3) \;=\; -0.39$$

in design units, giving natural units of $-0.39 \times 1 = -0.39cm$.

We predict five steps along the path of steepest ascent will bring us near the optimum region followed by a deterioration in response.

	Factors			
Step	A	B	(F)	Time
Base	8.5	4.00	(2.0)	12.99
Δ = step size	1.0	-0.39		
Base + 1Δ	9.5	3.61	(2.0)	15.22
Base + 2Δ	10.5	3.22	(2.0)	16.34
Base + 3Δ	11.5	2.83	(1.5)	**18.78**
Base + 4Δ	12.5	2.44	(1.5)	17.39
Base + 5Δ	13.5	2.05	(1.2)	7.24

3.3. PHASE 2: Determining Optimal Conditions: Second-order Response Surface

Near the optimum, the response surface is curved.

Stationary point has derivatives all 0, $\frac{\partial \hat{y}}{\partial x_i} = 0$, for $i = 1, \ldots, k$.

Plot surface or contour plots if k is 1 or 2.

Eigenvalues $\lambda_1, \ldots, \lambda_k$ and eigenvectors of $\hat{\mathbf{B}}$.

$$\hat{y} = b_0 + \underset{\sim}{x}'\underset{\sim}{b} + \underset{\sim}{x}'\hat{\mathbf{B}}\underset{\sim}{x}$$

$$\hat{\mathbf{B}} = \begin{bmatrix} b_{11} & b_{12}/2 & \cdots & b_{1k}/2 \\ & b_{22} & \cdots & b_{2k}/2 \\ & & \ddots & \vdots \\ \text{sym.} & & & b_{kk} \end{bmatrix}$$

Eigenvalues gives direction and magnitude of curvature.

Wanting a maximum, we hope all our λs are negative.

Central Composite Design (CCDs)

Workhorses of second-order response surface models are **central composite designs**.

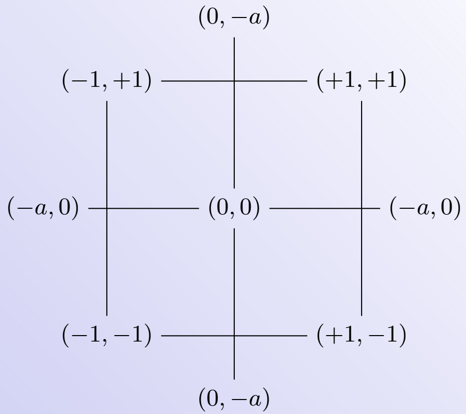
Composed of:

(fractional) factorial design for estimating main effects and two-factor interactions

$2k$ **axial points** $(\pm a, 0, \ldots, 0), \ldots, (0, \ldots, \pm a)$ for estimating the second-order terms

repeated center points second-order terms and internal estimate of (pure) error

The value of a can be 1, $\sqrt{k}$, or other values based on an optimality criteria.



At the top of our steepest ascent path we want to explore the space for curvature.

If curvature is not detected, we repeat steepest ascent.

Factor	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$
A	10.08	10.50	11.50	12.50	12.91
B	2.28	2.44	2.83	3.22	3.38
C			2.0		
D			1.5		
E			6.0		
F			1.5		
G			light		
H			against		

Obs	Ord	A	B	Time
1	7	-1	-1	13.65
2	3	$+1$	-1	13.74
3	11	-1	$+1$	15.48
4	5	$+1$	$+1$	13.53
5	9	0	0	17.38
6	2	0	0	16.35
7	1	0	0	16.41
8	10	$+1.414$	0	12.51
9	4	-1.414	0	15.17
10	6	0	$+1.414$	14.86
11	8	0	-1.414	11.85

Testing for response curvature, estimated standard error of $\bar{y}_F - \bar{y}_C$ is

$$\hat{\sigma}_{\bar{y}_F - \bar{y}_C} = \sqrt{0.3342\left(\frac{1}{4} + \frac{1}{3}\right)} = 0.4415 \quad (6)$$

and the critical cutoff at the 95% level ($L = 0.95$) is

$$\hat{\sigma}_{\bar{y}_F - \bar{y}_C} t_{c-1,(1+L)/2} = 0.4415(2.92) = 1.289 < |\bar{y}_F - \bar{y}_C| = 2.6133 \quad (7)$$

providing sufficient evidence that there is surface curvature.

On to second-order modeling.

The estimated second-order model is

$$\begin{aligned}\hat{y} &= b_0 + b_1 x_1 + b_2 x_2 + b_{12} x_{12} + b_{11} x_{11}^2 + b_{22} x_{22}^2 \\ &= 16.713 - 0.702 x_1 + 0.735 x_2 - 0.510 x_{12} - 1.311 x_{11}^2 - 1.554 x_{22}^2. \qquad (8)\end{aligned}$$

Residual	df	SS	MS	F Value	Pr $> F$
Lack of Fit	3	1.826	0.609	1.82	0.3737
Pure Error	2	0.668	0.334		
Total Error	5	2.495	0.499		

Table 3: Lack-of-Fit Test does not indicate the model does not fit.

Parameter	df	Estimate	Std Err	t Value	Pr $> \|t\|$
Intercept	1	16.71	0.408	40.98	0.0001
$x_1 = a$	1	-0.70	0.250	-2.81	0.0374
$x_2 = b$	1	0.73	0.250	2.94	0.0322
$x_{11} = a * a$	1	-1.31	0.297	-4.41	0.0070
$x_{12} = a * b$	1	-0.51	0.353	-1.44	0.2083
$x_{22} = b * b$	1	-1.55	0.297	-5.23	0.0034

Table 4: Second-order response surface parameter estimates show all but the crossproduct significant.

Model Assumptions:

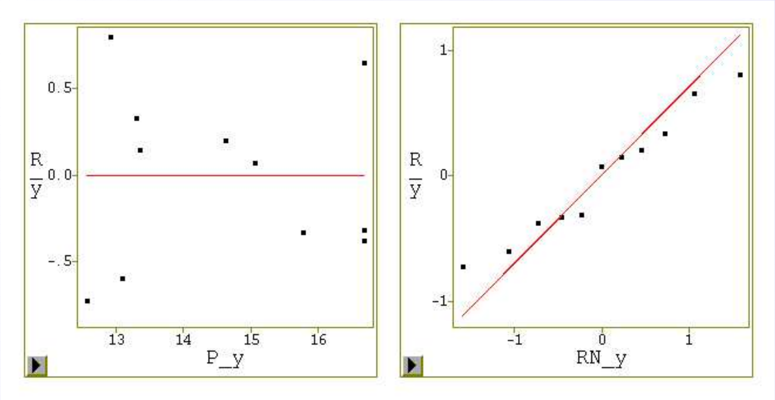


Figure 3: Residual plots does not suggest departures from normality.

Additionally, the Shapiro-Wilk normality test has a large p-value $= 0.7613$

Our model $R^2 = 0.92$, indicating the model should give precise predictions.

Stationary point: maximum?

SAS's `PROC RSREG` (response surface regression) gives

- location of stationary point $(a = -0.32, b = 0.29)$
- predicted response $\hat{y} = 16.9$
- the eigenvalues $(\lambda_a, \lambda_b) = (-1.15, -1.71)$ indicates a maximum — yes!

Factor	Coded	Natural
A	-0.32	11.18
B	0.29	2.94
C		2.0
D		2.0
E		6.0
F		1.5
G	light	light
H	against	against

Table 5: Optimum helicopter factor levels.

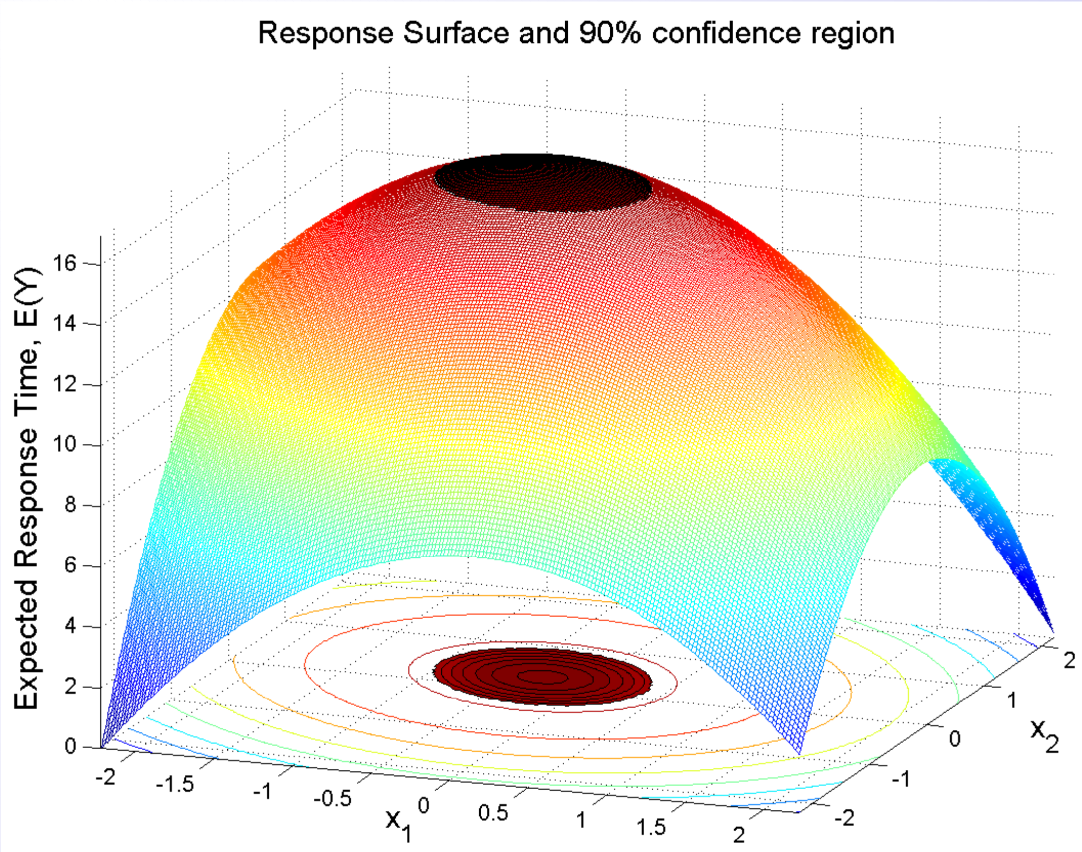
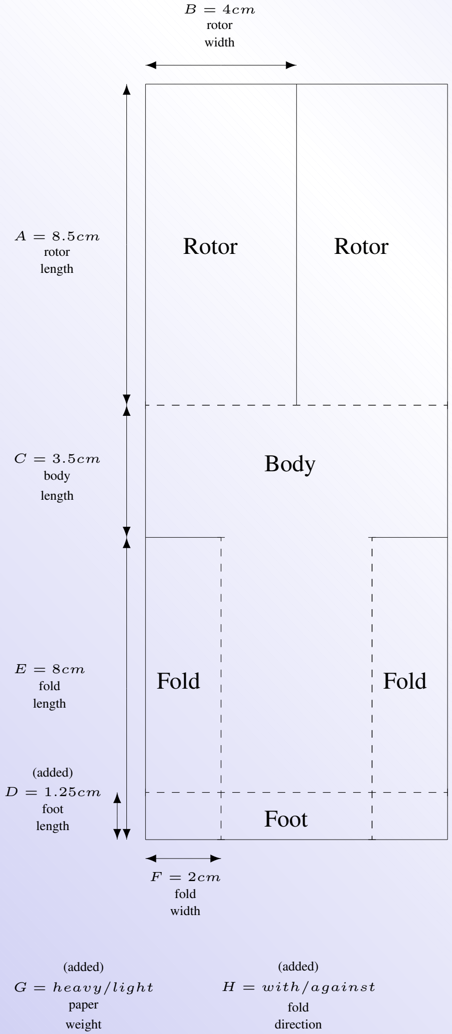
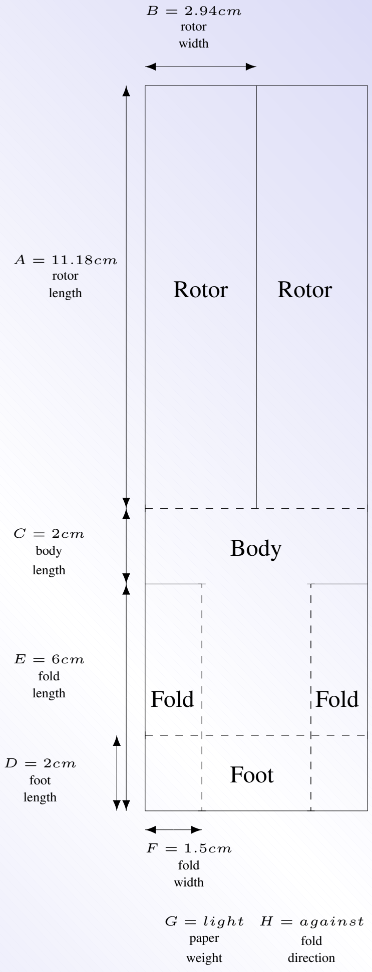


Figure 4: Estimated response surface with 90% confidence region.


$B = 4cm$
rotor
width
$A = 8.5cm$
rotor
length
Rotor
Rotor
$C = 3.5cm$
body
length
Body
$E = 8cm$
fold
length
Fold
Fold
(added)
$D = 1.25cm$
foot
length
Foot
$F = 2cm$
fold
width
(added)
$G = heavy/light$
paper
weight
(added)
$H = with/against$
fold
direction


$B = 2.94cm$
rotor
width
$A = 11.18cm$
rotor
length
Rotor
Rotor
$C = 2cm$
body
length
Body
$E = 6cm$
fold
length
Fold
Fold
$D = 2cm$
foot
length
Foot
$F = 1.5cm$
fold
width
$G = light$
paper
weight
$H = against$
fold
direction

To confirm the model prediction, a confirmatory experiment consisting of six new helicopters with the optimal setting are dropped. This confirmatory experiment gives a mean response of 17.81 with standard deviation 1.67, with a 95% confidence interval on the mean response of $(15.83, 18.04)$. Our model's predicted response $\hat{y} = 16.9$ falls within this interval, confirming the validity of our response model.

Obs	Time
1	15.54
2	16.40
3	19.67
4	19.41
5	18.55
6	17.29

A model fitting as well as ours, having a confidence region moderate in area, implies some flexibility in choosing factor levels for a near-optimum. In many situations, estimation of optimum conditions is very difficult. However, RSM should still produce important information about the process.

The methods of RSM are much broader than what appear in this paper, and with them almost any process can be better understood and improved. Besides, uncontrollable noise factors change with time, so a set of optimum conditions may be a fleeting concept. What is more important is for the analysis to reveal important information about the process and about the roles of the variables. The computation of a stationary point, a canonical analysis, or a ridge analysis may lead to important information about the process, and this in the long run will often be more valuable than a single set of coordinates representing an estimate of optimum conditions.

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↬ This 𝔡𝔬𝔠𝔲𝔪𝔢𝔫𝔱 was joyfully produced with LaTeX 2ε using the `pdfscreen.sty` package — and nothing else.

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