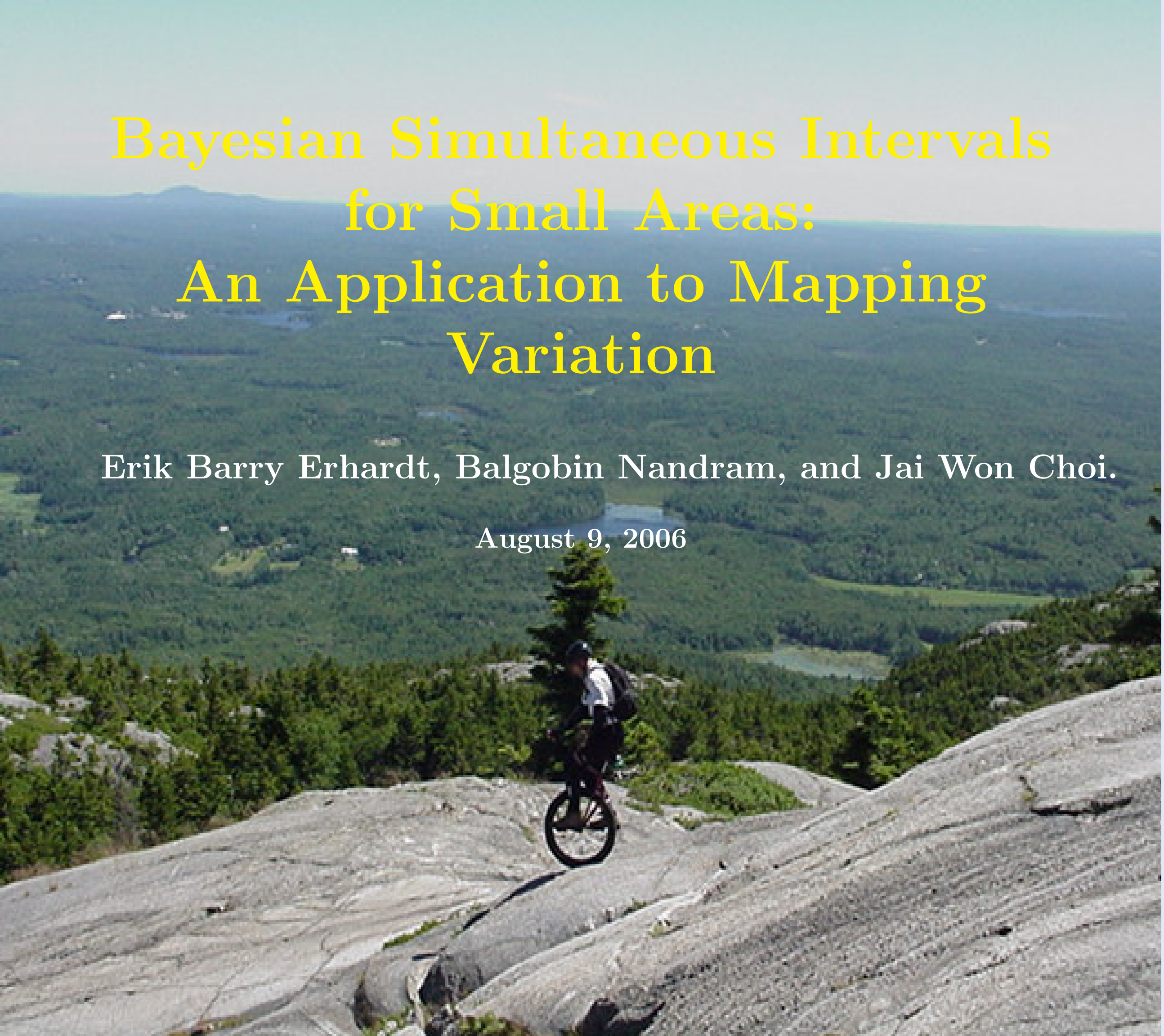

Bayesian Simultaneous Intervals for Small Areas: An Application to Mapping Variation

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1. Abstract

It is customary when presenting a choropleth map of rates or counts to present only the estimates (mean or mode) of the parameters of interest. While this technique illustrates spatial variation, it ignores the variation inherent in the estimates. We describe an approach to present variability in choropleth maps by constructing $100(1-\alpha)\%$ simultaneous intervals. The result provides three maps (estimate with two bands).

We propose two methods to construct simultaneous intervals from the optimal individual Highest Posterior Density (HPD) intervals to ensure joint simultaneous coverage of $100(1-\alpha)\%$.

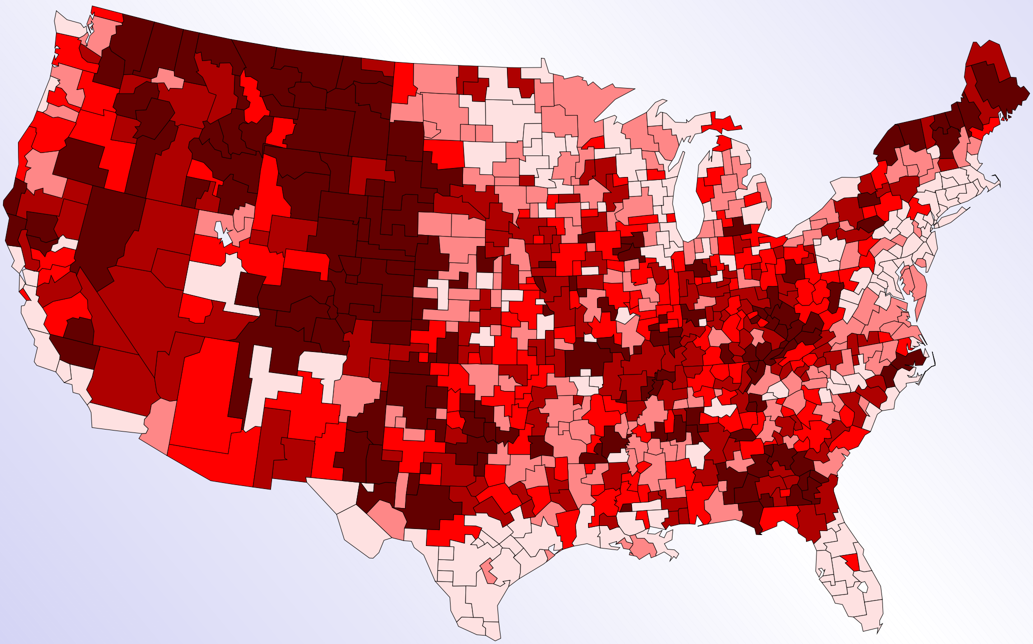
Both methods exhibit the main feature of multiplying the lower bound and dividing the upper bound of the individual HPD intervals by parameters $0 < \gamma_1, \gamma_2 < 1$ to "stretch" the interval until the simultaneous probability content is $100(1-\alpha)\%$. We employ the Nelder-Mead minimization algorithm to solve a system of non-linear equations involving the probability content and an optimality condition. Our Single-γ Method, where $\gamma_1 = \gamma_2$, optimizes over the probability content only, while the Double-γ Method includes an optimality condition. For our example, we found that these methods are comparable, appearing that the optimality condition adds very little information.

For illustrative purposes we apply our methods to chronic obstructive pulmonary disease (COPD) mortality rates from 1988–92, subset White Males age group 65 and older, for the continental United States for the 798 Health Service Areas (HSA).

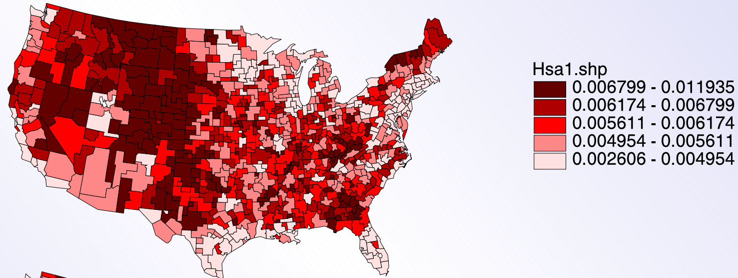
Mean Map for Age Classes 8, 9 and 10

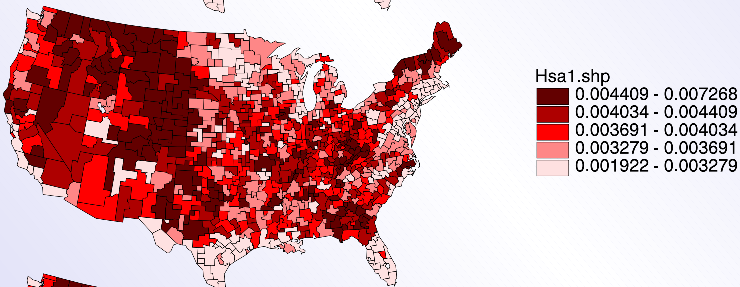
Deaths/1,000

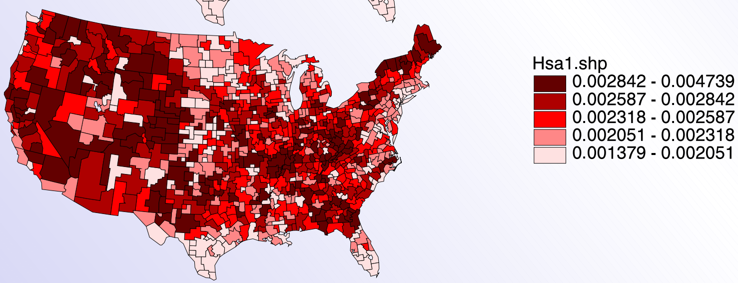
- 4.409 - 7.268
- 4.034 - 4.409
- 3.691 - 4.034
- 3.279 - 3.691
- 1.922 - 3.279



Simultaneous Interval Map
Age Classes 8, 9 and 10







1.1. Outline

1. Data, Literature Review.
2. Interval Estimation: CI, HPD, Simultaneous.
3. Poisson-Gamma Hierarchical Regression Model.
4. Maps (Regions), Tables.
5. Conclusions.

1.2. Source of Data

- Chronic Obstructive Pulmonary Disease (COPD) and its constituent diseases (asthma, chronic bronchitis and emphysema) and potential risk factors.
- All United States death certificates 1988–1992 and population census data for 1990.
- Number of HSAs by $\ell = 798$.
- Ensemble of mortality rate parameters $\underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda_\ell)'$.
- Deaths $\underset{\sim}{d} = (d_1, \ldots, d_\ell)'$ and population sizes $\underset{\sim}{n} = (n_1, \ldots, n_\ell)'$.

1.3. Methods for constructing simultaneous $100(1-\alpha)\%$ intervals

- Stretching.
- Twice as many unknowns as equations, not unique solution, computationally demanding.

$$\int_{a_\ell}^{b_\ell} \cdots \int_{a_1}^{b_1} \pi(\lambda_1, \ldots, \lambda_\ell | \underset{\sim}{d}) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_\ell = 1 - \alpha \quad (1)$$

$$\pi(a_1 | \underset{\sim}{d}) = \pi(b_1 | \underset{\sim}{d})$$

$$\vdots$$

$$\pi(a_\ell | \underset{\sim}{d}) = \pi(b_\ell | \underset{\sim}{d}) \quad (2)$$

2. Single-γ Method

- "Stretching" factor $0 < \gamma < 1$ on interval (a_i, b_i)

$$\int_{\gamma a_i}^{b_i/\gamma} \pi(\lambda_i | \, d_i) \, \mathrm{d}\lambda_i \;\; \geq \;\; \int_{a_i}^{b_i} \pi(\lambda_i | \, d_i) \, \mathrm{d}\lambda_i \qquad (3)$$

- Optimized HPD, stretch until desired content obtained, for some posterior distribution $\pi(\lambda_1, \ldots, \lambda_\ell | \, \underset{\sim}{d})$.

$$\int_{\gamma a_\ell}^{b_\ell/\gamma} \cdots \int_{\gamma a_1}^{b_1/\gamma} \pi(\lambda_1, \ldots, \lambda_\ell | \, \underset{\sim}{d}) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_\ell \;\; = \;\; 1 - \alpha \qquad (4)$$

2.1. Single-γ Method Computations

$\min_\gamma F(\gamma)$ attains zero, unique solution exists.

$$F(\gamma) \;\; = \;\; \left| \int_{\gamma a_\ell}^{b_\ell/\gamma} \cdots \int_{\gamma a_1}^{b_1/\gamma} \pi(\lambda_1, \ldots, \lambda_\ell | \, \underset{\sim}{d}) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_\ell - (1 - \alpha) \right| \qquad (5)$$

Theorem 2.1 *Given a set of HPD credible intervals* $A_i' = \{\lambda_i : a_i < \lambda_i < b_i\}$, $i = 1, \dots, \ell$, *defined on a set of densities with positive support and given the transformation* $A_i = \{\lambda_i : \gamma a_i < \lambda_i < b_i/\gamma\}$, *there is a unique* γ *satisfying* $P(\cap_{i=1}^{\ell} A_i | \underset{\sim}{d}) = 1 - \alpha$, *where* $\underset{\sim}{d}$ *is the data.*

Proof Outline

1. $\gamma = 1$, $P(A_i' | \underset{\sim}{d}) = 1 - \alpha$; $P(\cap_{i=1}^{\ell} A_i | \underset{\sim}{d}) < 1 - \alpha$.
2. $\gamma = 0$, $P(A_i' | \underset{\sim}{d}) = P(\cap_{i=1}^{\ell} A_i | \underset{\sim}{d}) = 1$.
3. As $\gamma \to 0$, $P(\cap_{i=1}^{\ell} A_i | \underset{\sim}{d})$ increases smoothly from a value less than $1 - \alpha$ to 1.

■

3. Double-γ Method

- "Stretching" factors $0 < \gamma_1 < 1$ and $0 < \gamma_2 < 1$ on interval (a_i, b_i).
- Lead to different values of $\{\gamma_1, \gamma_2\}$, content (6) takes precedence.

$$\int_{\gamma_1 a_\ell}^{b_\ell/\gamma_2} \cdots \int_{\gamma_1 a_1}^{b_1/\gamma_2} \pi(\lambda_1, \ldots, \lambda_\ell | \underset{\sim}{d}) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_\ell = 1 - \alpha \quad (6)$$

$$\pi(\gamma_1 a_i | \, d_i) = \pi(b_i/\gamma_2 | \, d_i) \quad (7)$$

3.1. Double-γ Method Computations

$\min_{\gamma_1,\gamma_2} F(\gamma_1, \gamma_2)$ approaches zero.

$$F(\gamma_1, \gamma_2) = \left| \int_{\gamma_1 a_\ell}^{b_\ell/\gamma_2} \cdots \int_{\gamma_1 a_1}^{b_1/\gamma_2} \pi(\lambda_1, \ldots, \lambda_\ell | \underset{\sim}{d}) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_\ell - (1 - \alpha) \right|$$

$$+ \ell^{-1} \sum_{i=1}^{\ell} |p(\gamma_1 a_i | \, d_i) - p(b_i/\gamma_2 | \, d_i)| \quad (8)$$

4. Poisson-Gamma Hierarchical Regression Model

$$
\begin{aligned}
d_i \mid \lambda_i &\overset{ind}{\sim} \text{Poisson}(n_i \lambda_i) \\
\lambda_i \mid \alpha, \underset{\sim}{\beta} &\overset{ind}{\sim} \text{Gamma}\left(\alpha, \alpha e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}}\right) \\
\pi(\alpha) &= \frac{1}{(1+\alpha)^2}, \qquad \alpha \geq 0 \\
\underset{\sim}{\beta} &\sim \text{Normal}\left(\mu_{\underset{\sim}{\beta}}, \Delta_{\underset{\sim}{\beta}}\right) \\
p(\underset{\sim}{\lambda}, \alpha, \underset{\sim}{\beta} \mid \underset{\sim}{d}) &\propto \prod_{i=1}^{\ell} \frac{\lambda_i^{d_i} e^{-n_i \lambda_i}}{d_i!} \\
&\times \prod_{i=1}^{\ell} \frac{\left(\alpha e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}}\right)^{\alpha} \lambda_i^{\alpha-1} \exp\left\{-\left(\alpha e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}}\right) \lambda_i\right\}}{\Gamma(\alpha)} \\
&\times \frac{1}{(1+\alpha)^2} \\
&\times \exp\left\{-\frac{1}{2}(\underset{\sim}{\beta} - \mu_{\underset{\sim}{\beta}})' \Delta_{\underset{\sim}{\beta}}^{-1} (\underset{\sim}{\beta} - \mu_{\underset{\sim}{\beta}})\right\} \qquad (9)
\end{aligned}
$$

5. Map Construction

$$p(\underset{\sim}{\lambda}|\, \underset{\sim}{d}) = \int_\Omega p(\underset{\sim}{\lambda}|\, \underset{\sim}{d}, \Omega)\pi(\Omega|\, \underset{\sim}{d})\, \mathrm{d}\Omega, \qquad \Omega = (\alpha, \underset{\sim}{\beta})$$

5.1. Constructing the Mean Map

- Rao-Blackwellized estimators for the λ_i, $\hat{\mu}_i$.
- Observed mortality rate $r_i = d_i/n_i$, $i = 1, \ldots, \ell$.
- $\Lambda_i = n_i/(n_i + \alpha e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}})$

$$\begin{aligned} \mathrm{E}(\lambda_i|\, \alpha, \underset{\sim}{\beta}, d_i) &= \Lambda_i r_i + (1-\Lambda_i) e^{\underset{\sim}{x}_i' \underset{\sim}{\beta}}. \\ \mu_i &= \mathrm{E}(\lambda_i|\, d_i) \\ &= \mathrm{E}_{\Omega|\, d_i} \left\{ \Lambda_i r_i + (1-\Lambda_i) e^{\underset{\sim}{x}_i' \underset{\sim}{\beta}} \right\}. \\ \hat{\mu}_i &\approx M^{-1} \sum_{h=1}^{M} \left\{ \Lambda_i^{(h)} r_i + (1-\Lambda_i^{(h)}) e^{\underset{\sim}{x}_i' \underset{\sim}{\beta}^{(h)}} \right\} \quad (10) \end{aligned}$$

- Posterior mean map, $\hat{\mu}_i$, $i = 1, \ldots, \ell$.

5.2. Constructing the Credible Interval Map

- Sort $\lambda_i^{(1)}, \ldots, \lambda_i^{(M)}$, $i = 1, \ldots, \ell$, giving $\lambda_i^{(1^*)}, \ldots, \lambda_i^{(M^*)}$.
- Choose the $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$ quantiles.
- For $\alpha = 0.05$ and $M = 10000$, choose the 0.025 and 0.975 quantiles, or $\lambda_i^{(250^*)}$ and $\lambda_i^{(9751^*)}$.

5.3. Constructing the HPD Interval Map

- Calculate ordinates $p(\lambda_i|\, d_i)$ at each $\lambda_i^{(1)}, \ldots, \lambda_i^{(M)}$, $i = 1, \ldots, \ell$.
- Find lower and upper bounds a_i and b_i, satisfying both conditions (11) and (12).

$$\int_{a_i}^{b_i} p(\lambda_i|\, d_i)\, \mathrm{d}\lambda_i = 1 - \alpha \tag{11}$$

$$p(a_i|\, d_i) = p(b_i|\, d_i) \tag{12}$$

$$\begin{aligned} f(a_i, b_i) &= \left| M^{-1} \sum_{h=1}^{M} \left\{ F(b_i|\, d_i, \Omega^{(h)}) - F(a_i|\, d_i, \Omega^{(h)}) \right\} - (1-\alpha) \right| \\ &\quad + \left| M^{-1} \sum_{h=1}^{M} p(a_i|\, d_i, \Omega^{(h)}) - M^{-1} \sum_{h=1}^{M} p(b_i|\, d_i, \Omega^{(h)}) \right| \end{aligned} \tag{13}$$

5.4. Constructing the Simultaneous Interval Map

5.5. Single-γ Method

$$f(\gamma) = \left| M^{-1} \sum_{h=1}^{M} \left[\prod_{i=1}^{\ell} \{ F(b_i/\gamma |\, d_i, \Omega^{(h)}) - F(\gamma a_i |\, d_i, \Omega^{(h)}) \} \right] - (1-\alpha) \right|$$

5.6. Double-γ Method

$$\begin{aligned} f(\gamma_1, \gamma_2) &= \left| M^{-1} \sum_{h=1}^{M} \left[\prod_{i=1}^{\ell} \{ F(b_i/\gamma_2 |\, d_i, \Omega^{(h)}) - F(\gamma_1 a_i |\, d_i, \Omega^{(h)}) \} \right] - (1-\alpha) \right| \\ &+ \ell^{-1} \sum_{i=1}^{\ell} \left| M^{-1} \sum_{h=1}^{M} \{ p(\gamma_1 a_i |\, \Omega^{(h)}) - p(b_i/\gamma_2 |\, \Omega^{(h)}) \} \right| \end{aligned}$$

Table 1: Single-γ Method values for γ.

Region		ℓ	γ	min(Lower)	max(Upper)
All	—	798	0.7711593	0.001379	0.011935
New England	1	23	0.9356322	0.002206	0.006979
Middle Atlantic	2	49	0.9192618	0.001713	0.006132
S. Atlantic-North	3	38	0.9109113	0.001852	0.007940
S. Atlantic-South	4	88	0.8659650	0.001568	0.008893
E. S. Central	5	88	0.8959517	0.002458	0.008035
E. N. Central	6	121	0.8849109	0.002182	0.006640
W. N. Central-North	7	45	0.8949770	0.002249	0.006502
W. N. Central-South	8	105	0.8395072	0.001970	0.007899
W. S. Central	9	115	0.8682747	0.001715	0.007171
Mountain-South	10	40	0.8657598	0.001768	0.009689
Mountain-North	11	38	0.8992667	0.003469	0.006927
Pacific	12	48	0.9094602	0.002098	0.005930

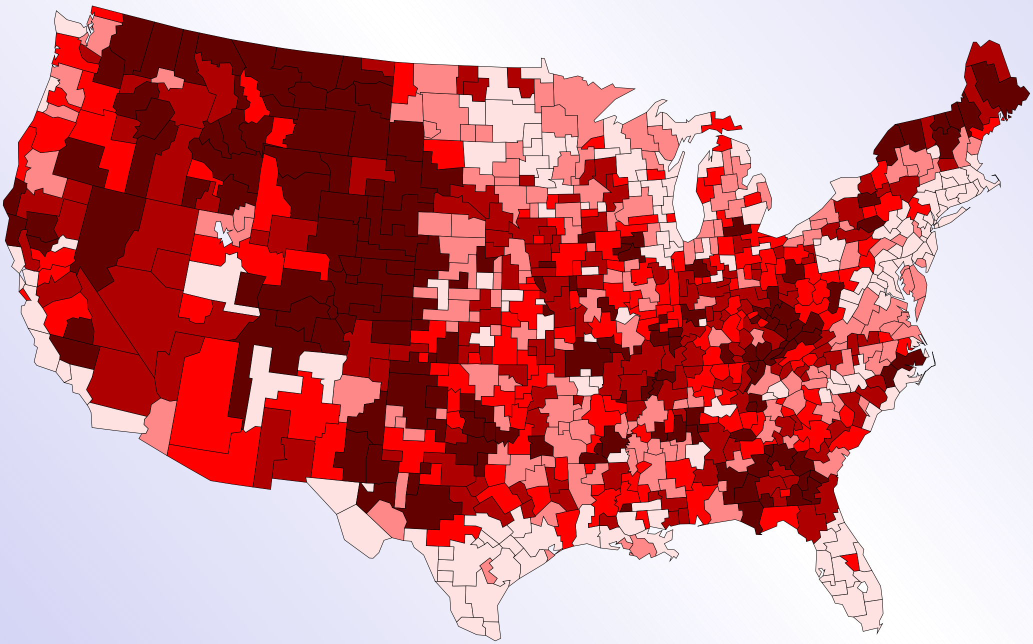
6. Cholopleth Maps

- The maps in this section are visualizations of the rate parameter, $\underset{\sim}{\lambda}$, for COPD mortality of White Males age classes 7, 8 and 9 (65 years and older) for each HSA.

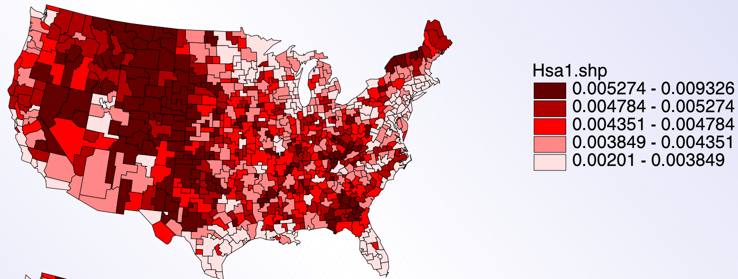
Mapping standards

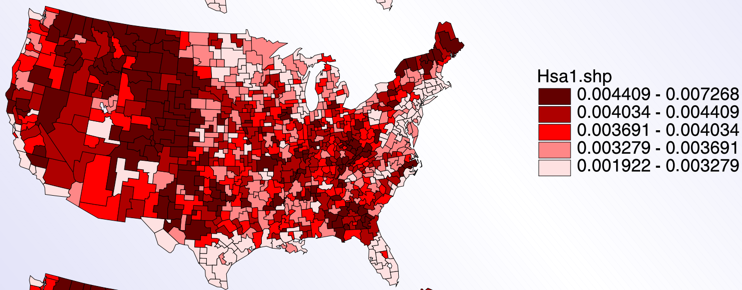
- Cutpoint, 95% intervals.
- Mean map between upper and lower maps for reference.

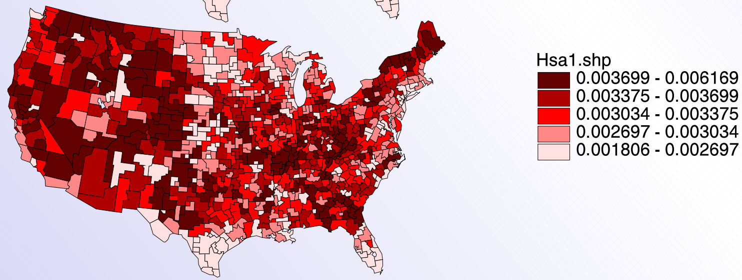
Mean Map for Age Classes 8, 9 and 10



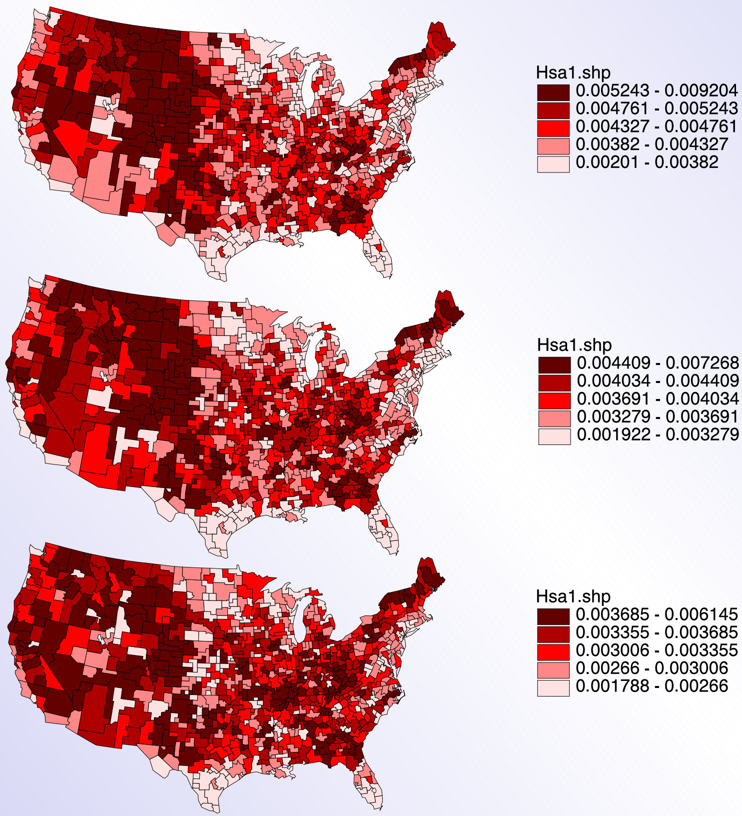
Credible Interval Map for Age Classes 8, 9 and 10







HPD Interval Map for Age Classes 8, 9 and 10



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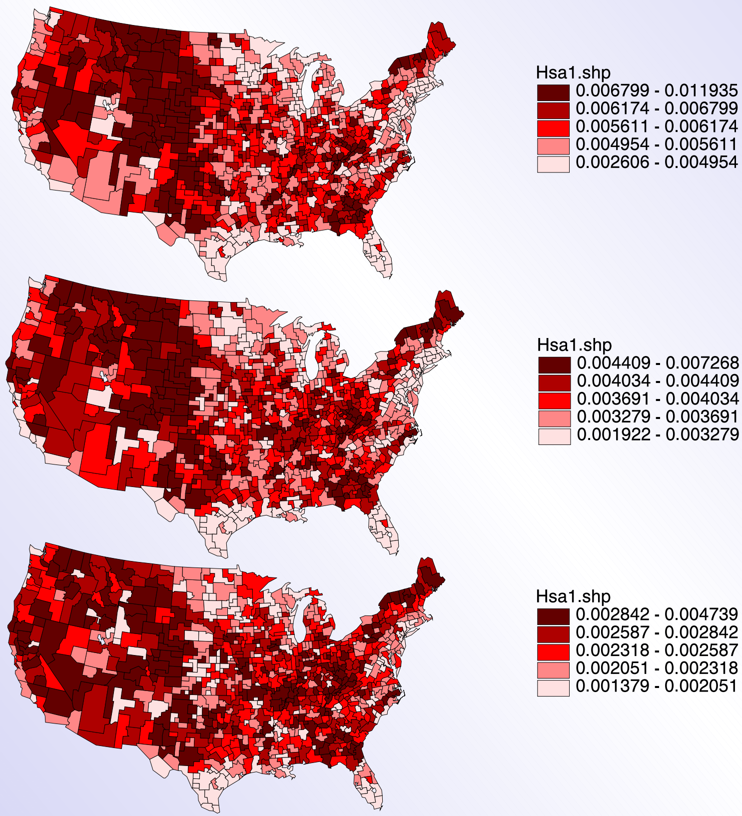
Simultaneous Interval Map
Age Classes 8, 9 and 10



Color Difference Map between High and Low Interval Maps for Credible, HPD and Simultaneous Age Classes 8, 9 and 10

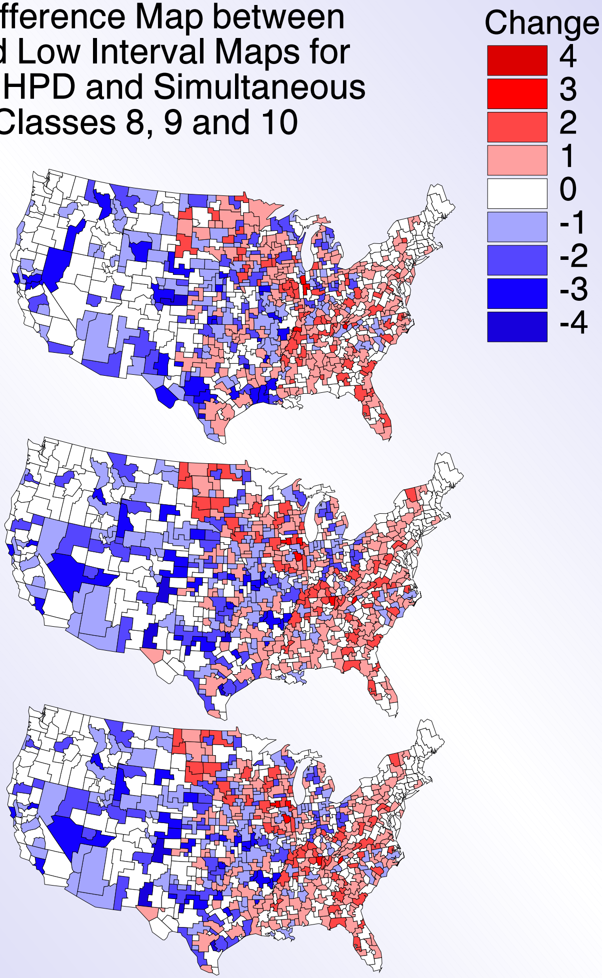


Table 2: Individual Legend Credible Interval difference between lower and upper bound maps.

Lower Quintile	Upper Quintile					
	1	2	3	4	5	
1	89	49	19	4	0	161
2	37	40	53	29	0	159
3	19	33	32	58	19	161
4	13	22	27	41	55	158
5	3	15	29	27	85	159
	161	159	160	159	159	798

Table 3: Individual Legend HPD Interval difference between lower and upper bound maps.

Lower Quintile	Upper Quintile					
	1	2	3	4	5	
1	93	46	17	5	0	161
2	34	43	55	27	0	159
3	18	31	34	57	21	161
4	12	22	28	40	56	158
5	5	16	26	30	82	159
	162	158	160	159	159	798

Table 4: Individual Legend Simultaneous Interval difference between lower and upper bound maps.

Lower Quintile	Upper Quintile					
	1	2	3	4	5	
1	93	46	17	5	0	161
2	34	43	55	27	0	159
3	18	31	34	57	21	161
4	13	22	28	40	56	159
5	4	16	26	30	82	158
	162	158	160	159	159	798

Table 5: Difference between individual legend lower and upper bound maps.

Method	Reg	-4	-3	-2	-1	0	1	2	3	4
CI		3	28	70	124	287	215	67	4	0
HPD		5	28	66	123	292	214	65	5	0
SγM	All	4	29	66	123	292	214	65	5	0
SγM	1	0	0	0	2	19	2	0	0	0
SγM	2	0	0	1	8	31	8	1	0	0
SγM	3	0	0	2	4	25	6	1	0	0
SγM	4	0	2	5	13	46	15	7	0	0
SγM	5	0	0	5	18	44	14	6	1	0
SγM	6	0	0	7	22	59	30	3	0	0
SγM	7	0	0	0	10	26	8	1	0	0
SγM	8	0	1	9	28	38	11	16	2	0
SγM	9	0	0	3	29	54	23	6	0	0
SγM	10	0	3	2	8	12	9	6	0	0
SγM	11	0	0	1	8	19	10	0	0	0
SγM	12	0	1	2	10	19	15	1	0	0

Note: SγM denotes Single-γ Method and Reg the region. The difference presented is the Lower map's color number subtracted from the Upper map's color number.

6.1. Conclusions

1. Simultaneous inference is a general statistical problem.
2. Our methodology provides a general multiple comparison solution to problems involving a large number of parameters.

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6.2. Simultaneous interval visualization example

- An extremely simple example with two independent Gamma distributions.

$$\lambda_1 | d_1 \sim \text{Gamma}(3, 1) \quad (14)$$
$$\lambda_2 | d_2 \sim \text{Gamma}(10, 0.5) \quad (15)$$

$$1 - \alpha = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \pi(\lambda_1, \lambda_2 | \underset{\sim}{d}) \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \quad (16)$$
$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \pi(\lambda_1 | d_1) \pi(\lambda_2 | d_2) \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \quad (17)$$
$$= \prod_{i=1}^{2} \left\{ \int_{a_i}^{b_i} \pi(\lambda_i | d_i) \, \mathrm{d}\lambda_i \right\}. \quad (18)$$

- $100(1-\alpha)^{1/2}\%$ HPD intervals, $(a_1, b_1), (a_2, b_2)$ in Figure 8.
- Simultaneous interval $\{(a_1, b_1), (a_2, b_2)\}$ in Figure 9 where the volume under the colored portion of the plot has probability $1 - \alpha$.

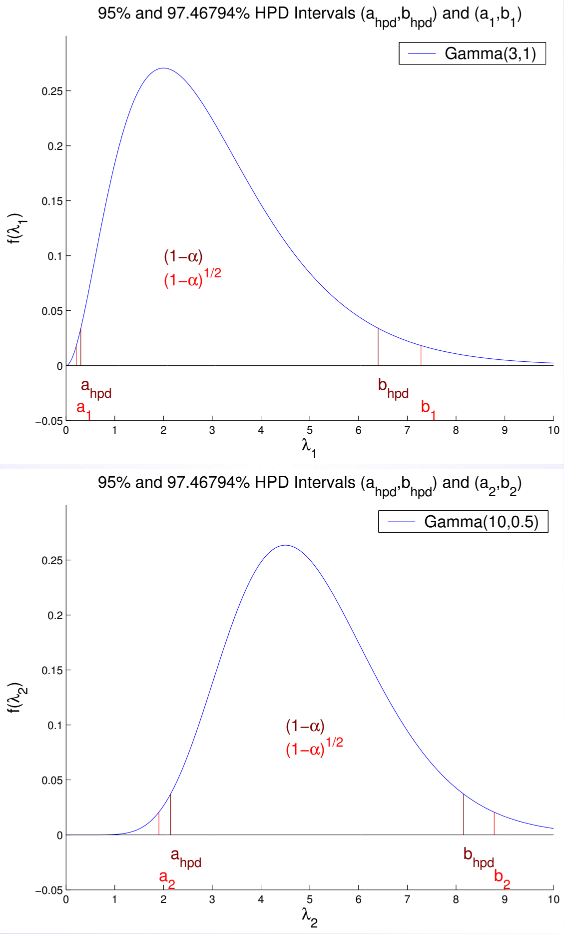

Figure 8: $100(1-\alpha)^{1/2}\%$ Individual HPD Intervals $(a_1, b_1), (a_2, b_2)$.

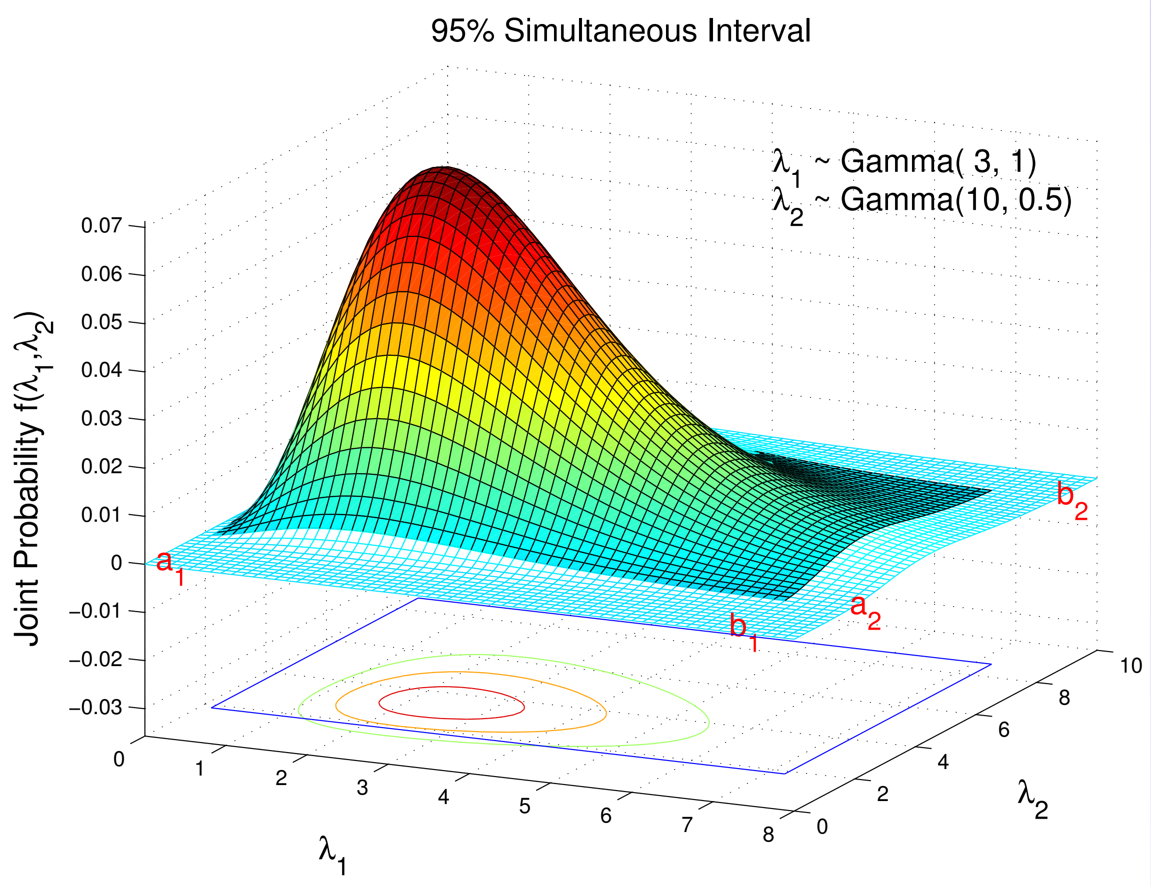

Figure 9: $100(1-\alpha)\%$ Simultaneous Interval $\{(a_1, b_1), (a_2, b_2)\}$.

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